False Advertising

Andrew Rhodes
Toulouse School of Economics

Chris M. Wilson
School of Business and Economics
Loughborough University

CCP Working Paper 15-13 v3
(updated October 2017)

Abstract

There is widespread evidence that some firms use false advertising to overstate the value of their products. Using a model in which a policymaker is able to punish such false claims, we characterize a natural equilibrium in which false advertising actively influences rational buyers. We analyze the effects of policy under different welfare objectives and establish a set of demand and parameter conditions where policy optimally permits a positive level of false advertising. Further analysis considers some wider issues including the implications for product investment and industry self-regulation.

Contact Details:

Andrew Rhodes	andrew.rhodes@tse-fr.eu
Chris M. Wilson	c.m.wilson@lboro.ac.uk

ISSN 1745-9648

False Advertising

Andrew Rhodes and Chris M. Wilson*

October 24, 2017

Abstract

There is widespread evidence that some firms use false advertising to overstate the value of their products. We consider a model in which a policymaker is able to punish such false claims. We characterize an equilibrium where false advertising actively influences rational buyers, and analyze the effects of policy under different welfare objectives. We establish precise conditions where policy optimally permits a positive level of false advertising, and show how these conditions vary intuitively with demand and market parameters. We also consider the implications for product investment and industry self-regulation, and connect our results to the literature on demand curvature.

Keywords: Misleading Advertising; Product Quality; Pass-through; Self-Regulation

JEL codes: M37; L15; D83

*Rhodes: Toulouse School of Economics, University of Toulouse Capitole, France; andrew.rhodes@tse-fr.eu. Wilson: School of Business and Economics, Loughborough University, UK; c.m.wilson@lboro.ac.uk.
We are thankful to the Editor, David Martimort, and two anonymous referees. We also thank Alexei Alexandrov, Mark Armstrong, Mike Baye, Joao Correia-da-Silva, Daniel Garcia, Andy Gavill, Matthew Gentzkow, Thomas Jeitschko, Justin Johnson, Bruno Jullien, Martin Peitz, Michael Powell, Robert Ritz, Jean Tirole, Tianle Zhang, and various audiences including those at the North American Econometric Society (Philadelphia), CEPR (Zurich), ESSET (Gerzensee), IIOC (Boston), EEA (Mannheim), CREST, HECER, THEMA, Toulouse, the Berlin IO Day, the 7th Workshop on the Economics of Advertising and Marketing (Vienna), EARIE (Munich), the NIE Workshop on Advertising (Manchester), and the University of East Anglia. We also thank Kamya Buch for her research assistance. Rhodes acknowledges financial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 670494).

1 Introduction

Buyers are often reliant on firms to obtain information about product characteristics. To exploit this, some firms deliberately engage in what we call 'false advertising' - the use of incorrect or exaggerated product claims. They do this in a range of different contexts, and despite potential legal penalties. Recent policy cases include Dannon which paid \$21m to 39 US states after it misled consumers about the health benefits of its Activia yogurt products; Skechers which paid \$40m after falsely stating that its toning shoes helped with weight loss; and Kellogg's which paid \$5m for wrongly claiming that its breakfast cereals enhanced childrens' immune systems. Similarly, in some related examples, car manufacturers such as Volkswagen, Hyundai and Kia have incurred a series of multi-million dollar penalties after cheating tests in order to make false claims about their emission levels or fuel efficiency.[1] Additional evidence also comes from academic research, which carefully documents the existence of false advertising and its ability to increase demand.[2]

However, despite this, the theoretical literature has largely ignored false advertising. In this paper, we develop a model where false advertising arises in equilibrium and actively influences rational buyers. Tougher legal penalties reduce the frequency of false adverts, but also increase their credibility. As a result of the latter effect, we show when and how stronger penalties can reduce buyer and social welfare. In particular, by using some results on demand curvature, the paper derives precise conditions on demand and market parameters such that a policymaker optimally uses a low penalty to permit a strictly positive level of false advertising. We then consider several wider issues including investment incentives and industry self-regulation.

In more detail, Section 2 introduces our main model where a monopolist is privately informed about its product quality. While we later extend the results in a number of ways, we initially focus on the case where quality is either 'high' or 'low' and where the two types have symmetric marginal costs. The policymaker first commits to a penalty for false advertising. Then having learned its type, the firm chooses a price and a (possibly false) claim about its quality. Buyers subsequently update their beliefs and make their purchase decisions, before the policymaker instigates any penalties. We believe this set-up closely approximates many important markets where buyers are unable to verify claims, or can only do so after a long time, and where policy plays a key role in regulating advertising.

Section 3 characterizes an equilibrium with some appealing economic properties, in which the high type advertises truthfully but the low type may engage in false advertising. This equilibrium does not resemble that of a standard price signaling game due to the assumption

[1] For further details, see https://www.ftc.gov/enforcement/cases-proceedings, goo.gl/yw437p; goo.gl/J1yJPx, goo.gl/7GWYsz. Accessed 10/23/17.

[2] E.g. Zinman and Zitzewitz (2016), Rao and Wang (2016), and Cawley *et al* (2013). See also Mayzlin *et al* (2014) for false advertising in the form of fake user reviews.

of symmetric costs. Instead, it smoothly unifies several otherwise separate cases depending on the level of the penalty. Firstly when the penalty is large, there is no false advertising. Here, as in the standard disclosure literature, quality claims are verifiable and product quality is perfectly revealed within a full separating equilibrium. Secondly when the penalty is small, the low type always conducts false advertising within a full pooling equilibrium. In this case, advertising is cheap talk and so buyers simply maintain their prior beliefs. Finally, when the penalty is moderate, our equilibrium involves a novel form of partially verifiable advertising. Here, the low type mixes between i) pooling with the high type by engaging in false advertising with a relatively high price, and ii) advertising truthfully with a relatively low price. Therefore when buyers observe a high claim, they positively update their belief that quality is high. Hence, in contrast to full separation, false advertising does arise in equilibrium but unlike in full pooling, advertised claims actively inflate buyers' beliefs beyond their priors even when false.

Section 4 analyzes how marginal changes in the penalty affect a variety of welfare measures. We first consider buyer surplus. Here, a reduction in the penalty increases the probability of false advertising and generates two opposing effects. The first 'persuasion' effect harms buyers by prompting them to buy too many units at an inflated price. The second effect derives from the impact of false advertising on damaging the credibility of claims. This issue goes back to at least Nelson (1974) and is well-documented empirically with studies showing how deception lowers credibility and reduces buyers' purchase intentions (e.g. Darke and Richie 2007, Newell *et al* 1998). However, rather than viewing this impact as detrimental, we document a beneficial 'price' effect whereby false advertising counteracts monopoly power by lowering buyers' quality expectations and prompting lower prices.

To compare these two effects under a relatively general form of demand, we utilize some recent results on demand curvature and cost pass-through (Weyl and Fabinger 2013, Anderson and Renault 2003). Such results are being used in an increasing number of applications, such as price discrimination (e.g. Chen and Schwartz 2015, Cowan 2012). However, rather than focusing on cost changes, we analyze the impact of changes in quality on price, which we term as 'quality pass-through'. In many cases, such as when the low quality product is particularly damaging or unsafe, we show how the persuasion effect dominates such that buyer surplus is maximized by eliminating false advertising. However, we also formalize a set of parameter conditions where the price effect dominates. Here, the optimal penalty is softer so as to induce a positive level of false advertising. Such cases include markets that are otherwise 'healthy' - where product quality levels are relatively high or where the probability of high quality is large. For instance, consider a recent case where the manufacturer of Nurofen was fined 6m Australian dollars (AUD) after marketing four specific painkillers at twice the price of their normal product despite the products being effectively identical. Although estimates suggested that this strategy yielded AUD45m, the firm was only fined

AUD6m. However, within the case, it was recognized that the strategy did not cause any physical harm. Hence, depending upon the exact parameters, our analysis could suggest that such a moderate fine may have indeed been optimal.[3]

Next, we analyze the effects of marginal penalty changes on profits and total welfare. Ex ante, the monopolist weakly prefers strong penalties that eradicate false advertising, because this enables it to extract more surplus from buyers when its quality is high. Hence, if the monopolist could commit, its choice of penalty would coincide with that preferred by buyers in many circumstances. However, for other cases, in contrast to the view that self-regulation is too soft, the monopolist's preferred penalty is actually too strong relative to buyers' preferences. We then consider total welfare. On the one hand, false advertising lowers credibility and so prompts any type with a high claim to further reduce its output below the socially desirable level. On the other hand, false advertising allows the low type to expand its output. We show that the latter dominates in otherwise 'healthy' markets, such that a positive level of false advertising is again optimal.

Section 5 extends the main model to the case where product quality is endogenous. This is important because false advertising can reduce product quality investment by limiting the available returns from high quality products. However, while we confirm that such an 'investment' effect prompts weakly higher penalties, we find that a positive level of false advertising can remain optimal for buyer and social welfare. In addition, once quality is endogenous, we show how a penalty increase can raise the probability of false advertising.

In most of the paper we restrict attention to equilibria in which the firm signals high quality through its report but not its price. In Section 6 we dispense with this restriction. We first show that common equilibrium refinements have little bite in our game due to the assumption of symmetric marginal costs. Second, to allow the refinements to have more power, we then introduce a small asymmetry in the types' marginal costs. A Pareto-based criterion uniquely selects a different but related equilibrium where price signaling may arise, and where a less rich form of our results and intuition still remains.

Finally, Section 7 shows how the results of the main model are robust to i) an arbitrary number of quality types, and ii) an alternative, multiplicative form of utility.

Background and Related Literature: In the US, most federal-level false advertising regulation is conducted by the FTC with various public measures, including monetary penalties. The FTC Deception Policy Statement defines a practice as deceptive if it is likely to misleadingly influence a reasonable consumer.[4] In Europe, most countries employ varying levels of industry self-regulation alongside statutory regulations under the EU Directive on Unfair

[3]See http://www.bbc.co.uk/news/world-australia-38337217. Accessed 10/23/17.

[4]For all policy documents listed in this paragraph see goo.gl/jL5Hrd, goo.gl/XBMkhL, and goo.gl/rVJ4F8. Accessed 10/23/17.

Commercial Practices. For instance, in the UK, most initial regulation is conducted by the industry-led Advertising Standards Authority, which is endorsed by various governmental bodies with the power to issue fines and bring criminal proceedings. Under the Directive, a practice is regarded as misleading if it is likely to deceive the average consumer and prompt them to make a transaction they would not otherwise make. Thus, like many other countries, the regulation of false advertising in the US and EU focuses only on the potential for consumer harm (under what we term as the 'persuasion' effect) and makes no account for any possible countervailing benefits (as consistent with our 'price' effect). This stance contrasts to that held by the FTC in relation to unfair practices, where a practice is deemed unfair if it causes a substantial injury to consumers that is not outweighed by any offsetting benefits to consumers or competition. Our paper suggests that this more balanced stance may also be sensible when regulating false advertising.

Within the advertising literature, attention is typically focused on truthful advertising (e.g. Anderson and Renault 2006, Johnson and Myatt 2006, and the reviews by Bagwell 2007, and Renault 2015). However, false advertising was discussed in some early work. For instance, contrary to our paper, Beales *et al* (1981, p.532) assert that "Because literally false statements offer no benefits to consumers, there is no reason to allow them". In other early work, Nelson (1974) provides a seminal discussion about the credibility of false advertising, noting "Deception requires not only a misleading or untrue statement, but somebody ready to be misled by the statement" (p.749). He then suggests that moderate regulation of false advertising may actually provide it with a source of credibility and enhance the incentives for firms to use it.

Since then, false advertising and its regulation are only beginning to be understood more formally in a small, recent literature. Some papers assume that buyers are naive and so believe all claims (e.g. Glaeser and Ujhelyi 2010, Hattori and Higashida 2012). Here, false advertising raises firms' profits, lowers consumer surplus, and can increase total welfare by offsetting the output distortion from imperfect competition. Other papers seek to endogenize advertising credibility with rational buyers. For example, some papers introduce heterogeneous tastes so that claims can gain credibility by forfeiting revenues from some buyers (e.g. Chakraborty and Harbaugh 2014). Another set of papers examine legal penalties in ways more related to our paper. Corts (2013, 2014a, 2014b) considers a setting where a monopolist knows its type but finds it costly to learn its precise product quality. Total welfare is assumed to be increasing in the fineness of consumer information. For low learning costs, total welfare is maximized with high penalties that induce the firm to learn its quality and eradicate false advertising. However when learning costs are higher, such that the firm chooses not to know its quality, lower penalties can raise welfare by inducing the firm to use speculative quality claims to signal its type even when such claims are false with positive probability. Under a different mechanism, Piccolo *et al* (2015) demonstrate that false advertising can maximize buyer surplus in a duopoly where the firms have different product qualities. They show that it is always optimal to use zero 'laissez-faire' penalties to induce full pooling rather than full separation. Intuitively, this creates a downward competitive pressure on prices by making the firms appear undifferentiated. However, after studying a monopoly version of their model, the authors suggest that this result can only arise in a competitive context.[5]

Our research differs from such work in a number of important respects. Firstly, we analyze a richer class of semi-pooling equilibria. Unlike the full pooling and separating equilibria in past models, these equilibria allow for false claims to arise in equilibrium, and to inflate buyers' beliefs beyond their priors. Secondly, by using a more general form of demand, we highlight a novel role for demand curvature. This enables us to provide precise conditions for when false advertising is beneficial, and to show how the optimal penalty varies intuitively with demand and market parameters. Thirdly, we show that competition is not necessary for false advertising to maximize buyer surplus. Moreover, we provide a unified set of results for buyer surplus, total welfare and firm profits.

Finally, our paper is related to some broader areas. First, aside from false quality claims, as considered in this article, other work is also beginning to understand alternative forms of false advertising, such as deceptive price claims. For instance, Armstrong and Chen (2017) examine firms' incentives to exaggerate their reported initial price when making price discounts, and show that a move from zero to full regulation need not improve welfare. More broadly, our article adds to the growing literature on consumer protection policy which also considers wider topics, such as high-pressure sales tactics (Armstrong and Zhou 2016) and refund rights (Inderst and Ottaviani 2013). Second, our model relates to some recent work on equilibrium lying and persuasion under full rationality (e.g. Kartik 2009, and Kamenica and Gentzkow 2011). In contrast, we study policy-related lying costs within a specific advertising context, where a third-party influences not only the amount of information that is communicated to buyers but also indirectly the price that they pay. Third, our article is related to the literatures on price signaling and quality disclosure (see Bagwell 2007 and Dranove and Jin 2010 for respective reviews). These literatures note that welfare can be maximized by some form of pooling due to either i) the output distortions from price signaling under separation, or ii) positive costs of disclosure. Instead, our results provide clear conditions under which (partial) pooling can be buyer- or welfare-optimal even when truthful disclosure is costless, and when full separation does not involve output distortions.

[5]In more distant work, Daughety and Reinganum (1997) study how punitive damages for false safety claims can help support separating equilibria when firms face liability losses for product defects, Barigozzi *et al* (2009) examine false comparative advertising, and Drugov and Troya-Martinez (2015) analyze false advice where firms can also choose the vagueness of their claims.

2 Model

A monopolist sells one product to a unit mass of potential buyers. The monopolist is privately informed about its product quality q. Specifically, the product is of low quality L with probability $x \in (0,1)$, and of high quality H with probability $1-x$, where $-\infty < L < H < \infty$. Average ex ante quality is then defined as $\bar{q} = xL + (1-x)H$. For our main analysis we assume that marginal costs are independent of quality and normalized to zero. Each buyer has a unit demand and values a given product of quality q at $q+\varepsilon$, where ε is a buyer's privately known match with the product. This match is drawn independently across buyers using a distribution function $G(\varepsilon)$ with support $[a,b]$ where $-\infty \leq a < b \leq \infty$. The associated density $g(\varepsilon)$ is strictly positive, continuously differentiable, and has an increasing hazard rate.

The monopolist sends a publicly observable advertisement or 'report' $r \in \{L, H\}$ at no cost, where a report $r = z$ is equivalent to a claim "Product quality is z". The binary report space is without loss because there are only two firm types and reports are costless. A policymaker is able to verify any advertised claim and impose a penalty ϕ if it is false, where false advertising is defined as the use of a high quality report $r = H$, by a firm with low quality $q = L$.[6] The policymaker can costlessly choose any level of penalty, $\phi \geq 0$, in order to maximize one of three possible objectives: buyer surplus, total profit, or total welfare. Any penalties that involve a fine go to the policymaker.

The timing of the game is as follows. At stage 1 the policymaker publicly commits to a penalty ϕ for false advertising. At stage 2 the monopolist privately learns its quality, q. Given q, it then announces a price $p(q)$ and issues a report $r(q) \in \{L, H\}$. At stage 3 buyers decide whether to buy the product taking into account ϕ as well as the firm's price and report. Finally at stage 4 the policymaker verifies the advertised claim and administers the penalty, ϕ, if it is false. The solution concept is Perfect Bayesian Equilibrium (PBE). All omitted proofs are included in the appendix unless stated otherwise.

3 Equilibrium Analysis

3.1 Benchmark with Known Quality

As a first step, consider a benchmark case in which the firm is known to have quality q. Quality claims are then redundant because it is weakly optimal for the firm to use truthful advertising. An individual buyer purchases the product if and only if $\varepsilon \geq p - q$ such that demand equals $D(p-q) = 1 - G(p-q)$. The firm then chooses its price to maximize

[6] One can interpret ϕ as an expected penalty if claims are only verified probabilistically, in which case ϕ is the product of the probability of detection and level of penalty (conditional on detection). More broadly, ϕ can also include any direct costs of making a false claim.

$p\left[1-G\left(p-q\right)\right]$, and so:

Lemma 1. *Suppose the firm is known to have quality* q, *and define* $\underset{\sim}{q}=-b$ *and* $\tilde{q}=-a+1/g(a)$. *The firm's optimal price,* $p^{*}(q)$, *is increasing in* q *and satisfies:*

$$p^{*}(q)=\begin{cases}0 & \text{if } q\leq \underset{\sim}{q}\\ \frac{1-G(p^{*}(q)-q)}{g(p^{*}(q)-q)} & \text{if } q\in\left(\underset{\sim}{q},\tilde{q}\right)\\ a+q & \text{if } q\geq\tilde{q}\end{cases} \tag{1}$$

When $q\leq\underset{\sim}{q}$, quality is so low that the firm would make zero sales even if it priced at marginal cost. The market is inactive, and we normalize the firm's price to zero without loss. When instead $q\in(\underset{\sim}{q},\tilde{q})$, the firm optimally sells to some but not all buyers such that $p^{*}\left(q\right)$ satisfies the usual first order condition. Finally if $q\geq\tilde{q}$, quality is so high that the firm optimally sells to all potential customers by pricing at the willingness-to-pay of the marginal buyer, $a+q$, such that the market is 'covered'. However, for some distributions, $\tilde{q}=\infty$ and so this final case is redundant. Henceforth to avoid some uninteresting cases, let $\bar{q}>\underset{\sim}{q}$ (or $\bar{q}+b>0$) such that a product of average quality always has some positive value.

In our later analysis, it will be important to understand how the optimal price varies with quality, and we will sometimes refer to the derivative $p_{q}^{*}(q)$ as 'quality pass-through'. To this end, it is useful to define demand curvature as $\sigma(\psi)=-[1-G(\psi)]g'(\psi)/g(\psi)^{2}$ i.e. the elasticity of the slope of inverse demand (see Aguirre *et al* 2010 and Weyl and Fabinger 2013).[7] When $\sigma(\psi)\leq 1-\rho$ demand is also said to be ρ-concave (see Anderson and Renault 2003). We now show that the level of quality pass-through, $p_{q}^{*}(q)$, depends upon the value of q. Firstly for $q\in(\underset{\sim}{q},\tilde{q})$, after differentiating the first order condition:

$$p_{q}^{*}(q)=\frac{1-\sigma(p^{*}(q)-q)}{2-\sigma(p^{*}(q)-q)}. \tag{2}$$

Our assumption of an increasing hazard rate implies that $D(p-q)$ is logconcave in price. This guarantees that $\sigma(\psi)\leq 1$ (or equivalently, that demand is 0-concave) such that $p_{q}^{*}(q)\in[0,1)$. Intuitively, an increase in quality q produces a parallel outward shift in the inverse demand curve, and the firm optimally responds by both charging a higher price and by selling to strictly more buyers.[8] Secondly, where appropriate, when $q\geq\tilde{q}$ quality pass-through is one since the optimal price tracks the willingness-to-pay of the lowest buyer, $a+q$.

[7] In more detail, let Q denote output and $P(Q)$ be the inverse demand curve. The elasticity of the slope of inverse demand is then $-QP_{QQ}(Q)/P_{Q}(Q)$. After some standard manipulations this gives $D\cdot D_{pp}/(D_{p})^{2}$, which simplifies to the expression in the text.

[8] Weyl and Fabinger (2013) note that the optimal price change from any outward unit shift in inverse demand (such as an increase in quality within our model) equals one minus cost pass-through.

The equilibrium profit earned by a firm of known quality q can be written as

$$\pi^*(q) = p^*(q)\left[1 - G\left(p^*(q) - q\right)\right], \tag{3}$$

which is continuously differentiable for all $q \in (\underset{\sim}{q}, \tilde{q})$. It is straightforward to show that $\pi^*(q)$ is globally increasing and convex in q. Finally, buyer surplus can be expressed as

$$v^*(q) = \int_{p^*(q)}^{b+q} \left[1 - G\left(z - q\right)\right] dz, \tag{4}$$

which is also continuously differentiable for all $q \in (\underset{\sim}{q}, \tilde{q})$. Notice that $v^*(q)$ is i) strictly positive and strictly increasing in $q \in (\underset{\sim}{q}, \tilde{q})$, and ii) (where appropriate) constant in $q > \tilde{q}$. We further discuss the shape of $v^*(q)$ in Section 4.1 below.

3.2 Privately-Known Quality

Henceforth we assume that the firm is privately informed about its quality. As is typical in signaling games, there exists a large number of PBE because buyers can attribute any off-path claim or price to the low type. We therefore proceed by considering PBE under two restrictions.

Restriction 1: The high type makes a truthful claim with probability one, $r(H) = H$.

This allows us to focus on settings with the potential for meaningful false advertising by the low type. Further, notice that for given beliefs, the use of a high report is costless for the high type, but costly for the low type whenever $\phi > 0$. Therefore intuitively one would expect the high type to use a high report to signal its quality. See Daughety and Reinganum (1997) for a similar argument.

If the low type makes a false claim $r(L) = H$, it must charge the same price as the high type otherwise buyers would correctly infer its type. Therefore, we need to determine the price charged by the firm when it reports $r = H$. As we later explain, many such prices can form a PBE even after applying standard refinements due to the assumption of symmetric marginal costs. Hence, to select amongst these prices we impose a second restriction:

Restriction 2: On observing $r = H$, buyer beliefs are independent of prices.

Conditional on reporting $r = H$ and charging any price p, the high type earns exactly ϕ more than the low type because they both have the same marginal cost of production. Therefore the types' payoffs are not single-crossing in price, and so any attempt by the high type to signal its quality through price can be mimicked at exactly the same cost by the low type. As a result, the high type cannot profit from price signaling and so Restriction 2 suggests it is reasonable that buyers should not make additional inferences from the firm's price beyond those implied by the high report. See Corts (2013) for a related argument, and also Section 6 where the restriction is further discussed.[9]

To see the implications of Restriction 2, let $q_H^e \equiv E(q|r = H)$ denote buyers' beliefs following a high report. An individual buyer purchases the product if and only if $\varepsilon \geq p - q_H^e$, such that the firm's profit (before any penalties) is $p[1 - G(p - q_H^e)]$, and so it optimally charges $p^*(q_H^e)$.

We now show that Restrictions 1 and 2 lead to a unique equilibrium with some attractive economic properties and analytical features.

Proposition 1. *Under Restrictions 1 and 2 there exists a unique PBE*[10] *in which:*
i) A high type claims $r = H$ and charges $p^*(q_H^e)$.
ii) A low type randomizes. With probability y^* it claims $r = H$ and charges $p^*(q_H^e)$. With probability $1 - y^*$ it claims $r = L$ and charges $p^*(L)$.
- When $\phi \leq \phi_1 \equiv \pi^*(\bar{q}) - \pi^*(L)$, $y^* = 1$
- When $\phi \geq \phi_0 \equiv \pi^*(H) - \pi^*(L)$, $y^* = 0$
- When $\phi \in (\phi_1, \phi_0)$, $y^* \in (0, 1)$ and uniquely solves

$$\pi^*(q_H^e) - \phi = \pi^*(L), \tag{5}$$

$$\text{where} \quad q_H^e = \frac{xy^*L + (1-x)H}{1 - x + xy^*}. \tag{6}$$

iii) Buyer beliefs are $\Pr(q = H|r = L) = 0$ *and* $\Pr(q = H|r = H) = \frac{1-x}{1-x+xy^*}$.

The exact nature of the equilibrium depends intuitively and smoothly on whether the penalty is 'low', 'medium', or 'high'. Firstly, if $\phi \geq \phi_0$ the equilibrium has full separation, with both types reporting truthfully because the penalty is sufficiently high that truth-telling is a dominant strategy for the low type. Advertising is perfectly informative and claims are fully believed by buyers. Consequently, each type $q \in \{L, H\}$ sets its full information price $p^*(q)$, and earns its full information profit $\pi^*(q)$. Secondly, if $\phi \leq \phi_1$ the equilibrium has full pooling, with both types sending a high report because the penalty is sufficiently low that false advertising is a dominant strategy for the low type. Upon seeing a high claim buyers then maintain their prior belief that the firm's average quality is $\bar{q}$. As such, both types set

[9] An alternative model-based justification for Restriction 2 is as follows. In practice, the claim and pricing decisions may be taken by different divisions or managers within the firm. In particular, suppose at stage 2 of the game one division selects the report and then decides whether to inform the price-setter of the actual quality for some positive (but potentially small) communication cost. The (otherwise uninformed) price-setter then chooses a price. Informing the price-setter is a dominated strategy because once the report is made the payoffs of the two types are identical and so the optimal price does not depend on q. Therefore, in equilibrium, the price-setter remains uninformed. Moreover, to the extent that buyers do not put weight on strictly dominated strategies, they should believe that the price-setter is uninformed, and so should not use price to make additional inferences about quality.

[10] The equilibrium is unique up to off-path beliefs. Following an off-path report $r = L$ and price, there can be many beliefs which are consistent with our restrictions and which generate the same equilibrium outcome.

$p^*(\bar{q})$ and earn profits of $\pi^*(\bar{q})$ and $\pi^*(\bar{q}) - \phi$ respectively. Finally and most interestingly, if $\phi \in (\phi_1, \phi_0)$ the equilibrium is semi-pooling. Here the low type randomizes, making a false report with probability $y^* \in (0,1)$ and a truthful report with complementary probability $1 - y^*$. Upon seeing a high report buyers therefore update their belief about quality to $q_H^e \equiv E(q|r = H) \in (\bar{q}, H)$ as expressed in (6). Consequently the high type reports $r = H$ and charges $p^*(q_H^e)$, and the low type randomizes between $r = L$ and price $p^*(L)$, and report $r = H$ and price $p^*(q_H^e)$. The probability with which the low type lies, y^*, satisfies equation (5) and ensures that the payoffs from false advertising and truth-telling are equal. If instead the low type lied with a probability above (below) y^*, buyers' expected quality q_H^e would be too low (high) and hence the low type would have a strict preference to report truthfully (falsely), yielding a contradiction. Hence randomization with probability y^* is an essential feature of this equilibrium.

Notice that false advertising exists whenever $\phi < \phi_0$. However, for very low penalties, $\phi \leq \phi_1$, buyers effectively ignore high claims and simply maintain their prior belief within the pooling equilibrium. On the other hand, when $\phi \in (\phi_1, \phi_0)$, the semi-pooling equilibrium has the more appealing feature that high claims provide buyers with some useful information, and thus induce them to positively update their belief, with $q_H^e > \bar{q}$, even if the high claim turns out to be false.

4 The Effects of Policy

First consider the effects of policy on the level of false advertising, y^*. By using equations (5) and (6) it follows that:

Lemma 2. *The level of false advertising y^*, is continuous and weakly decreasing in the level of penalty ϕ.*

This feature of the model is useful analytically in the subsequent sections, and ensures that stronger policy smoothly increases the informativeness of advertising. When $\phi > \phi_0$ or $\phi < \phi_1$, a low quality firm has a strict preference for truth-telling or lying respectively, and so small changes in ϕ have no effect. However when $\phi \in [\phi_1, \phi_0]$, the probability of false advertising y^* satisfies the indifference condition (5) and is strictly decreasing in ϕ from 1 to 0. Intuitively, to maintain indifference of the low type as ϕ increases, high reports must become more credible. Since buyers are Bayesian, this is only possible if y^* is strictly lower.

4.1 Buyer Surplus

We now consider the effects of policy on a variety of welfare measures, starting with buyer surplus. Using Proposition 1 we can write expected buyer surplus as

$$E(v) = x(1-y^*)\int_{p^*(L)-L}^{b}(L+\varepsilon-p^*(L))\,\mathrm{d}G(\varepsilon) + xy^*\int_{p^*(q_H^e)-q_H^e}^{b}(L+\varepsilon-p^*(q_H^e))\,\mathrm{d}G(\varepsilon)$$
$$+(1-x)\int_{p^*(q_H^e)-q_H^e}^{b}(H+\varepsilon-p^*(q_H^e))\,\mathrm{d}G(\varepsilon). \tag{7}$$

In words, with probability $x(1-y^*)$ the firm sends a low report and charges $p^*(L)$. Buyers correctly infer low quality, buy if $\varepsilon \geq p^*(L)-L$, and receive $L+\varepsilon-p^*(L)$. Then with probability $1-x+xy^*$ the firm sends a high report and charges $p^*(q_H^e)$. Buyers update their beliefs according to equation (6), and buy if $\varepsilon \geq p^*(q_H^e)-q_H^e$. With conditional probability $xy^*/\,(1-x+xy^*)$, the product is low quality, and buyers receive $L+\varepsilon-p^*(q_H^e)$. With conditional probability $(1-x)/\,(1-x+xy^*)$, the product is high quality, and buyers receive $H+\varepsilon-p^*(q_H^e)$. After collecting terms and using the definition of $v^*(q)$ in equation (4), the above expression simplifies as follows, where $E(v)$ is just a convex combination of $v^*(L)$ and $v^*(q_H^e)$.

$$E(v) = x(1-y^*)v^*(L) + (xy^*+1-x)v^*(q_H^e). \tag{8}$$

We now exploit the smooth feature of our equilibrium to investigate the effect of a marginal increase in the penalty ϕ. In particular Lemma 3 will provide conditions under which an increase in the penalty ϕ leads to a reduction in expected buyer surplus. To understand why this can happen, recall that the level of false advertising y^* is a decreasing function of ϕ, and note that an increase in y^* produces two effects. On the one hand, buyers are more likely to receive a false advert and so be persuaded to buy a low quality product at an inflated price $p^*(q_H^e) > p^*(L)$. However on the other hand, the increase in lying damages the credibility of high quality claims, and forces any type making such a claim to reduce its price. In more detail, using equation (8) one can write

$$\frac{\partial E(v)}{\partial y^*} = \underbrace{x\left[v^*\left(q_H^e\right)-\left(q_H^e-L\right)D\left(p^*(q_H^e)-q_H^e\right)-v^*(L)\right]}_{\text{‘Persuasion’ effect}}$$
$$-\underbrace{(1-x+xy^*)D(p^*(q_H^e)-q_H^e)\cdot p_q^*(q_H^e)\cdot\frac{\partial q_H^e}{\partial y^*}}_{\text{‘Price’ effect}}. \tag{9}$$

The first term is a ‘persuasion’ effect. Conditional on the firm having low quality, a marginal increase in lying replaces the surplus that the buyer would have received if the firm had told the truth, $v^*(L)$, with the surplus associated with false advertising, $v^*\left(q_H^e\right) - \left(q_H^e - L\right) D\left(p^*(q_H^e) - q_H^e\right)$. To explain this latter surplus, note that after observing a high report, buyers update their beliefs to q_H^e, and expect to receive a surplus $v^*\left(q_H^e\right)$. However since quality is low, each of the $D\left(p^*(q_H^e) - q_H^e\right)$ units bought is worth $q_H^e - L$ less than anticipated. This harms buyers by prompting them to pay too much and to potentially buy too many units of a low quality product. The second term in (9) is a 'price' effect. A marginal increase in lying decreases the probability that a high claim is true, and so causes rational buyers to revise their belief q_H^e downwards. This lowers the level of monopoly power and induces any type with a high report to reduce its price by $-p_q^*(q_H^e) \cdot (\partial q_H^e / \partial y^*)$. Hence conditional on the firm sending a high report, buyer surplus is strictly higher on each of the $D(p^*(q_H^e) - q_H^e)$ inframarginal units bought. Importantly for our later results, this effect is more powerful when quality pass-through $p_q^*(q_H^e)$ is larger.

To establish whether the persuasion effect or the price effect dominates, it is useful to simplify equation (9) in the following way

$$\frac{\partial E(v)}{\partial y^*} = x\left[v^*\left(q_H^e\right) - v^*(L) - \left(q_H^e - L\right) v_q^*\left(q_H^e\right)\right]. \tag{10}$$

It is then clear that the shape of $v^*(q)$ determines whether expected buyer surplus is increasing or decreasing in y^*. Intuitively, a small increase in y^* leads to posterior beliefs which have the same mean but put more weight closer to the prior. Using Jensen's inequality the impact of this on expected buyer welfare depends on whether $v^*(q)$ is convex or concave. To proceed further we impose a regularity condition on $v^*(q)$. In particular, while our earlier restriction $\sigma(\psi) \leq 1$ ensures that $v^*(q)$ is monotonically increasing, Condition 1 now ensures that $v^*(q)$ is s-shaped:

Condition 1. There exists a threshold $\hat{q}_v \in (\underset{\sim}{q}, \infty)$ such that $v^*(q)$ is strictly convex for $q \in (\underset{\sim}{q}, \hat{q}_v)$. For $q > \hat{q}_v$ either i) $v_q^*(q) = 0$, or ii) $v^*(q)$ is strictly concave with $\lim_{q \to \infty} v_q^*(q) = 0$.

Condition 1 is satisfied by many common demand functions. Full details are provided in Section A of the Supplementary Appendix.[11] There we show that Condition 1 imposes a restriction on the slope of demand curvature and how it varies with quality. As such, it ensures that quality pass-through is relatively low when q_H^e is small, but is relatively high when q_H^e is large.

In more detail, Condition 1i is satisfied by any demand with constant curvature, a rich class which includes linear and exponential demand (see Bulow and Pfleiderer 1983). Here, $\hat{q}_v = \tilde{q} < \infty$. Intuitively, when $q < \tilde{q}$ small increases in q raise the surplus of inframarginal

[11] This appendix may also prove useful for other wider literatures. For instance, a recent literature on third-degree price discrimination uses a related restriction to ensure that buyer surplus is convex with respect to marginal cost (e.g. Chen and Schwartz 2015 and Cowan 2012).

buyers by a constant amount $1-p_q^*(q) > 0$, but also induce more and more buyers to purchase, and hence $v^*(q)$ is strictly convex. However once $q > \tilde{q}$ the market is covered and quality pass-through is one, $p_q^*(q) = 1$, and so any further increases in q leave $v^*(q)$ unchanged. Condition 1ii is satisfied by many demands with increasing curvature - including those derived from the Normal, Weibull, Power, Type I Extreme Value, and Logistic distributions. Here, $v^*(q)$ is strictly increasing for all $q > \underset{\sim}{q}$, implying that the market never becomes covered, $\tilde{q} = \infty$. Intuitively, when $q < \hat{q}_v$, quality pass-through is low and few buyers purchase: small increases in q mainly increase demand and so generate a convex $v^*(q)$. On the other hand when $q > \hat{q}_v$, quality pass-through is high and most buyers purchase: small increases in q mainly increase the surplus of inframarginal buyers, but by less and less because quality pass-through is increasing, and so this generates a concave $v^*(q)$.[12] We can then state:

Lemma 3. *Consider $\phi \in [\phi_1, \phi_0]$ and suppose that Condition 1 holds.*
i) If $L < \hat{q}_v$ expected buyer surplus is quasiconcave in ϕ. In particular there exists a threshold $q^*(L) \geq \hat{q}_v$ such that expected buyer surplus is strictly increasing in ϕ if $q_H^e < q^*(L)$, but strictly decreasing in ϕ if $q_H^e > q^*(L)$. The threshold $q^*(L)$ is weakly decreasing in L and satisfies $\lim_{L \to \hat{q}_v} q^*(L) = \hat{q}_v$.
ii) If $L > \hat{q}_v$ expected buyer surplus is weakly decreasing in ϕ.

Lemma 3 can be understood as follows. First consider $L < \hat{q}_v$. When $q_H^e < q^*(L)$ buyers are relatively pessimistic upon seeing a high report, quality pass-through is relatively weak, and so the persuasion effect dominates. When instead $q_H^e > q^*(L)$ buyers are relatively optimistic upon seeing a high report, quality pass-through is relatively strong, and so the price effect dominates. Hence a stronger policy benefits buyers in the former situation, but harms them in the latter. Second consider $L > \hat{q}_v$. If demand satisfies Condition 1i, the price and persuasion effects cancel such that $\partial E(v)/\partial \phi = 0$. Intuitively the market is fully covered irrespective of the firm's claim, and buyers pay either $a + L$ under a low claim, or $a + q_H^e$ under a high claim. The average price paid is therefore $a + \bar{q}$ which is independent of ϕ. However if instead demand satisfies Condition 1ii, the price effect strictly dominates such that $\partial E(v)/\partial \phi < 0$.

We now consider the optimal level of penalty, ϕ^*. To ease exposition, we henceforth focus on the (more interesting) case where $L < \hat{q}_v$. Recalling Lemma 3, we find that:

[12] Condition 1 is satisfied by all the closed-form, continuously differentiable, logconcave demands listed in Bagnoli and Bergstrom (2005). One interesting exception is homogeneous buyers, under which $v^*(q) = 0$ for all q, such that expected buyer surplus is independent of ϕ. Another exception is demand derived from the Laplace distribution. Here there is a threshold q_0 such that $v^*(q)$ is strictly convex for $q < q_0$, kinks downwards at $q = q_0$, but behaves like Condition 1ii for $q > q_0$. Numerical examples show that when $L < q_0 < H$ expected buyer surplus is not necessarily quasiconcave in y^*.

Proposition 2. *Fix $L < \hat{q}_v$ and suppose that Condition 1 holds. The buyer-optimal penalty, ϕ^*, is characterized as follows:*
i) When $H \le q^*(L)$, $\phi^* \ge \phi_0$ such that $y^* = 0$.
ii) When $\bar{q} < q^*(L) < H$, $\phi^* = \pi^*(q^*(L)) - \pi^*(L)$ such that $y^* = \frac{(H-q^*(L))(1-x)}{(H-q^*(L))(1-x)+q^*(L)-\bar{q}} \in (0,1)$.
iii) When $q^*(L) \le \bar{q}$, $\phi^* \le \phi_1$ such that $y^* = 1$.

Proposition 2 provides a range of demand and parameter conditions where a buyer-oriented policymaker refrains from eradicating false advertising. Recall from Lemma 3 that for $L < \hat{q}_v$, a marginal decrease in false advertising increases buyer surplus if and only if buyers are relatively pessimistic about high claims, with $q_H^e < q^*(L)$. Therefore when $H \le q^*(L)$ buyer surplus is globally decreasing in y^* and the policymaker optimally eliminates false advertising. However when $\bar{q} < q^*(L) < H$ buyer surplus is strictly quasiconcave and maximized at some $y^* \in (0,1)$, such that the optimal penalty tolerates some false advertising. Finally when $q^*(L) \le \bar{q}$ buyer surplus is globally increasing in y^* and so the policymaker fully permits false advertising.

The fact that a positive level of false advertising can generate a higher buyer surplus than under full information (where $y^* = 0$) gives several policy implications. First, any instinctive per se implementation of strong penalties or blanket prohibitions on false advertising may actually limit buyer surplus. Second, the optimal use of advertising penalties is superior to an outright ban on low quality products. Such a ban only generates a surplus $E(v) = (1-x)v^*(H)$, which is weakly less than the surplus under full information.

Finally, we further detail the conditions under which positive false advertising is optimal.

Corollary 1. *Given Condition 1 and $L < \hat{q}_v$, the buyer-optimal level of false advertising, y^*, is increasing in the product quality levels, L and H, and the (ex ante) probability that quality is high, $(1-x)$.*

Intuitively, when the monopolist's product quality technology is relatively 'healthy', the expected quality from a high claim, q_H^e, is relatively high such that the price effect becomes relatively more powerful. On the contrary, in less 'healthy' settings, the persuasion effect becomes especially harmful. Figure 1 illustrates these results in the simple case where buyer match values are uniformly distributed on $[0,1]$, and low quality is normalized such that $L = 0$. The figure plots the buyer-optimal y^* as a function of H, for three different values of $(1-x)$. In this example $q^*(L) = 1$, and hence for $H < 1$ it is optimal to use a tough policy to eliminate false advertising. However for $H > 1$ some false advertising is optimal, and consistent with Corollary 1 the optimal y^* is an increasing function of both H and $(1-x)$. To see the magnitude of these policy effects, consider a more specific example in which $x = 1/2$ and $H = 3/2$, where the buyer-optimal policy induces $y^* = 1/2$. Here, expected buyer surplus is 30% higher than it would be under a high penalty where $y^* = 0$, and approximately 6% higher than it would be under a low penalty where $y^* = 1$.

Figure 1: The Buyer-Optimal Level of False Advertising for Example Parameters

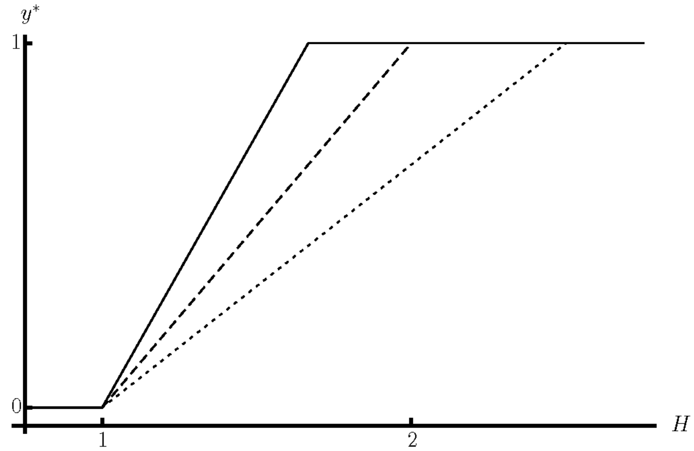

Figure 1 plots the buyer-optimal level of false advertising, y^*, as a function of H, for the case where $G(\varepsilon) = \varepsilon$ on $[0, 1]$, $L = 0$, and $(1-x)$ equals either 3/5 (the solid line), 1/2 (the dashed line), or 2/5 (the dotted line).

4.2 Profits

We now examine the effect of policy on profits. To begin, consider each individual firm type:

$$E(\pi_L) = \begin{cases} \pi^*(\bar{q}) - \phi & \text{if } \phi < \phi_1 \\ \pi^*(L) & \text{if } \phi \geq \phi_1 \end{cases} \quad \text{and} \quad E(\pi_H) = \begin{cases} \pi^*(\bar{q}) & \text{if } \phi < \phi_1 \\ \pi^*(L) + \phi & \text{if } \phi \in [\phi_1, \phi_0] \\ \pi^*(H) & \text{if } \phi > \phi_0 \end{cases} \tag{11}$$

This is explained as follows. When $\phi < \phi_1$ the equilibrium has full pooling such that each type earns $\pi^*(\bar{q})$, but the low type also incurs a penalty ϕ. When $\phi \in [\phi_1, \phi_0]$ the low type is indifferent between lying and truth-telling, and so earns $\pi^*(L)$. The high type, meanwhile, earns $\pi^*(q_H^e)$ which is equal to $\pi^*(L) + \phi$ from (5). Finally when $\phi > \phi_0$ the equilibrium has full separation, and so each type earns its full information payoff.

Remark 1. An increase in ϕ reduces $E(\pi_L)$, but increases $E(\pi_H)$.

Intuitively, stronger regulation increases the high type's payoff because it leads buyers to update more optimistically upon seeing a high claim. However tougher regulation hurts a low type because it becomes costlier to mimic a high type. Now consider expected equilibrium profit, $E(\Pi) = xE(\pi_L) + (1-x)E(\pi_H)$:

Proposition 3. *Expected profit is quasiconvex in ϕ and minimized at $\phi = \phi_1$. In addition:*
i) If $L < \tilde{q}$, expected profit is maximized by $\phi^* \geq \phi_0$.
ii) If $L \geq \tilde{q}$, expected profit is maximized by either $\phi^* = 0$ or $\phi^* \geq \phi_0$.

A small increase in regulation can either benefit or harm the monopolist, depending upon how existing regulation ϕ compares with ϕ_1. In addition, it is straightforward to see from (11) that $\phi \in (0, \phi_0)$ is strictly dominated under an expected profit objective. This implies that a positive penalty should never be paid in equilibrium. Then, given the convexity of $\pi^*(q)$, full separation with $\phi^* \geq \phi_0$ is always weakly optimal. Intuitively, strong regulation allows the firm to extract buyer surplus more effectively when it has high quality. Hence, if the monopolist could credibly commit to effective self-regulation (perhaps through a third party), it would weakly prefer to avoid using false advertising. In some circumstances, such as when $L < \hat{q}_v$ and $H < q^*(L)$, this self-regulation might be acceptable to buyers because the monopolist's preferred level of penalty coincides with that of the buyers. This may offer some support for Europe's industry-led regulation. However, in other circumstances self-regulation would go against buyers' preferences e.g. when $L < \hat{q}_v$ and $H > q^*(L)$. Here, contrary to any concerns that self-regulation may be too lax, the monopolist's preferred level of penalty is strictly higher than buyers'.

4.3 Total Welfare

We now consider total welfare. Suppose that the penalty, ϕ, is in the form of a fine which is as valuable to the policymaker as it is to the firm. Using Proposition 1, we can write expected total welfare as follows.

$$E(w) = x\,(1 - y^*)\,[v^*\,(L) + \pi^*\,(L)] + (1 - x + xy^*)\,[v^*\,(q_H^e) + \pi^*\,(q_H^e)]\,. \tag{12}$$

Notice that this expression is not just the summation of expected buyer surplus in (8), and weighted firm-type profits in (11), because by assumption the penalty has social value.

We will shortly provide conditions under which a stronger penalty ϕ reduces expected total welfare. We start with some intuition and then formally state the result. The firm uses its monopoly power to restrict output below the socially efficient level. A marginal increase in the level of false advertising y^* then changes this output distortion in two ways. First, it lowers the credibility of any high claim, and so forces any type with such a claim to further reduce its output below the socially optimal level. Second however, it also induces buyers to over-estimate a low type's quality, thereby causing the low type to increase its output. Under certain circumstances this latter output expansion can raise welfare and dominate the former effect.[13] In more detail:

$$\frac{\partial E(w)}{\partial y^*} = \underbrace{x\Big[v^*\left(q_H^e\right) - v^*\left(L\right) - (q_H^e - L)D(p^*(q_H^e) - q_H^e) + \pi^*(q_H^e) - \pi^*\left(L\right)\Big]}_{\text{Output expansion by a firm with } q=L}$$
$$+ \underbrace{(1-x+xy)D(p^*(q_H^e) - q_H^e)\left(1 - p_q^*(q_H^e)\right)\frac{\partial q_H^e}{\partial y^*}}_{\text{Output contraction by a firm with } r=H} \quad (13)$$

The first term in (13) represents the change in welfare when a low type moves from reporting $r = L$ and generating a total surplus of $v^*(L) + \pi^*(L)$, to claiming $r = H$ and generating a surplus of $v^*(q_H^e) - (q_H^e - L)\, D(p^*(q_H^e) - q_H^e) + \pi^*(q_H^e)$. This term is positive if and only if L is above a certain threshold. Intuitively, the low type's socially optimal output level $D(-L)$ is increasing in L. Moreover when a low type engages in false advertising, its output increases from $D\left(p^*(L) - L\right) \leq D(-L)$ to $D(p^*(q_H^e) - q_H^e)$. Therefore if L is relatively small, this 'output expansion' effect goes far beyond the efficient level and so is bad for welfare. However if L is relatively large, the output expansion effect brings the low type closer to the efficient level, and so is good for welfare. The second term in (13) represents the change in surplus generated by a firm that claims to have high quality, following a small increase in y^*. As explained above, this is unambiguously negative because an increase in y^* reduces the credibility (and hence output) of a firm that reports $r = H$. Important for our later results, this 'output contraction' effect tends to be smaller when quality pass-through $p_q^*(q_H^e)$ is larger since in that case the firm's output is less sensitive to buyers' belief about its quality.

To determine which of these two output effects dominates, it is useful to define $w^*(q) \equiv v^*(q) + \pi^*(q)$, and simplify (13) as follows

$$\frac{\partial E(w)}{\partial y^*} = x\left[w^*\left(q_H^e\right) - w^*(L) - \left(q_H^e - L\right) w_q^*\left(q_H^e\right)\right]. \quad (14)$$

There is a clear parallel with our earlier buyer surplus analysis. In particular by Jensen's inequality, the shape of $w^*(q)$ determines how an increase in y^* changes expected welfare. Therefore to proceed further, we impose an additional regularity condition:

Condition 2. There exists a threshold $\hat{q}_w \in (\underset{\sim}{q}, \infty)$ such that $w^*(q)$ is strictly convex for $q \in (\underset{\sim}{q}, \hat{q}_w)$. For $q > \hat{q}_w$ either i) $w_q^*(q) = 1$ with $\lim_{q\uparrow\hat{q}_w} w_q^*(q) > 1$, or ii) $w^*(q)$ is strictly concave with $\lim_{q\to\infty} w_q^*(q) = 1$.

This imposes a further restriction on the slope of demand curvature and how it varies with quality. While it differs from Condition 1, it plays a qualitatively similar role to

[13] The output distortions on the two types' output are reminiscent of the output effects in third-degree price discrimination (e.g. Aguirre *et al* 2010).

ensure that $w^*(q)$ is s-shaped. In particular it ensures that when q_H^e is small, quality pass-through is relatively low such that the output contraction effect is powerful, and that when q_H^e is large the opposite is true. Similar to earlier, part i) deals with situations where $\hat{q}_w = \tilde{q} < \infty$ such that the market can become covered, whilst part ii) deals with cases where $\tilde{q} = \infty$ such that the market is never fully covered. As detailed in Section A of the Supplementary Appendix, Condition 2 holds for all demands with constant curvature, and several others with increasing curvature including those derived from the Normal, Weibull and Power distributions.[14] Therefore, we can state:

Lemma 4. *Consider $\phi \in [\phi_1, \phi_0]$ and suppose that Condition 2 holds.*
i) When $q_H^e < \hat{q}_w$ expected total welfare is strictly increasing in ϕ.
ii) When $L < \hat{q}_w < q_H^e$ expected total welfare is quasiconcave in ϕ. In particular there exists a threshold $L^*(q_H^e) < \hat{q}_w$ such that expected total welfare is strictly increasing in ϕ if $L < L^*(q_H^e)$, but strictly decreasing in ϕ if $L > L^*(q_H^e)$. The threshold $L^*(q_H^e)$ is weakly decreasing in q_H^e.
iii) When $L > \hat{q}_w$ expected total welfare is weakly decreasing in ϕ.

Lemma 4 can be understood as follows. When $q_H^e < \hat{q}_w$ quality pass-through is relatively small, such that the output contraction effect dominates, and so $E(w)$ decreases in the level of false advertising y^*. When $L < \hat{q}_w < q_H^e$ quality pass-through is relatively stronger, and so the output contraction effect is weaker. A small increase in y^* therefore raises welfare provided L is sufficiently large, such that the expansion in the low type's output is not (too) excessive. Finally when L is large with $L > \hat{q}_w$, an increase in the penalty can never raise welfare as the output expansion always weakly dominates.

In a model with one firm type and naive buyers, Glaeser and Ujhelyi (2010) show that some false advertising always improves total welfare by increasing output towards the social optimum. Our welfare result with two types and rational buyers is more complex for the following reasons. First in our model, when the low type pools with the high type it increases its output to $D(p^*(q_H^e) - q_H^e)$, which may actually exceed the socially optimal level. Second in our model, false advertising reduces the credibility of high claims and so generates another welfare-reducing 'output contraction' effect which is not present in their paper.

Now consider the implications for the optimal penalty. To ease exposition, we focus on the (more interesting) case where $L < \hat{q}_w$. First, note that the policymaker will always eliminate false advertising with $\phi^* \geq \phi_0$ when $H < \hat{q}_w$. This follows from Lemma 4 because $q_H^e < \hat{q}_w$ for all $y^* \in [0, 1]$. Second, for the remaining cases, the policymaker is assumed to choose the lowest level of false advertising if indifferent. Proposition 4 then confirms that it

[14] Some other common distributions with increasing curvature do not satisfy Condition 2, such as the Logistic and Type I Extreme Value (Min). Here $w^*(q)$ is globally strictly convex, such that $E(w)$ is maximized by $y^* = 0$. Also, following from footnote 12, $E(w)$ is always weakly decreasing in y^* with homogeneous buyers, and $E(w)$ is not necessarily quasiconcave in y^* under the Laplace distribution.

can remain optimal to induce a positive level of false advertising:

Proposition 4. *Fix $L < \hat{q}_w < H$ and suppose that Condition 2 holds. The welfare-optimal penalty, ϕ^*, is characterized as follows:*
i) When $L \leq L^*(H)$, $\phi^* \geq \phi_0$ such that $y^* = 0$.
ii) When $L > L^*(H)$, $\phi^* < \phi_0$ such that $y^* \in (0, 1]$.

Hence, even though full separation with $\phi^* \geq \phi_0$ involves full information prices and no output distortions, it may still be optimal to induce false advertising to prompt some form of pooling. In line with intuition, one can show that the optimal level of false advertising is weakly lower than that under a buyer surplus objective. However, the optimal level of false advertising remains increasing in the 'healthiness' of the market (e.g. L, H, and $(1 - x)$).[15]

5 Endogenous Quality Investment

We now extend the main model to examine some additional effects of false advertising in a market with endogenous product quality. Such effects have been largely ignored within the literature. However they are important because the existence of false advertising may reduce the incentives to invest in product quality by limiting the credibility of advertising. Suppose that the firm is initially endowed with low quality L, but can upgrade to high quality H by paying an investment cost C. This cost is drawn privately from a distribution $F(C)$ on $(0, \infty)$, with corresponding density $f(C) > 0$. The move order is then as follows. At stage 1 the policymaker commits to a penalty ϕ. At stage 2 the firm learns its investment cost C, and privately chooses whether to upgrade. It also announces its report and price. The game then proceeds as in the main model, with buyers making their purchase decisions, and the policymaker instigating any potential penalties. Let $x^*(\phi)$ denote the endogenous probability that the firm has low quality.

There always exists a trivial equilibrium in which $x^*(\phi) = 1$. If buyers believe that product quality is low for all reports and prices, the firm has no incentive to invest. However, in general, there also exist other alternative PBE. Henceforth, as before, we impose Restrictions 1 and 2. Moreover whenever possible, we focus on equilibria where the firm invests with positive probability.

Lemma 5. *i) When $\phi = 0$ all equilibria have $x^*(\phi) = 1$. ii) When $\phi \in (0, \phi_0]$ there is a unique equilibrium (up to off-path beliefs) satisfying our restrictions, with $x^* = 1 - F(\phi) \in (0, 1)$.*

[15]Finally, false advertising can remain optimal even when a fraction τ of the penalty is 'lost' and does not contribute to total welfare. For instance when $L^*(\bar{q}) < L < \hat{q}_w \leq \bar{q}$ the optimal penalty induces $y^* = 1$ for all $\tau \in [0, 1]$. The only difference is that when $\tau = 0$ any $\phi \in [0, \phi_1]$ maximizes total welfare, whereas when $\tau > 0$ it is more attractive to reduce the penalties incurred, so $\phi^* = 0$ is the unique optimum.

Intuitively, an increase in ϕ induces investment by widening the gap in profits earned by high and low quality firms. In more detail, when $\phi = 0$ buyers cannot distinguish between high and low quality. The firm earns the same profit regardless and therefore chooses not to invest. Alternatively when $\phi \geq \phi_0$, claims are fully credible. A low quality firm reports $r = L$ and earns $\pi^*(L)$, whilst a high quality firm reports $r = H$ and earns $\pi^*(H)$. Since the gains from investing are $\pi^*(H) - \pi^*(L) \equiv \phi_0$, the firm upgrades if and only if $C \leq \phi_0$. Finally when $\phi \in (0, \phi_0)$, the level of false advertising is necessarily positive for the same reason as in the main model. This further implies that a high quality product earns ϕ more than a low quality product such that the firm invests with probability $F(\phi)$. However unlike the main model, the probability of false advertising y^* is not necessarily decreasing everywhere in ϕ. Recall the definition $q_H^e \equiv E(q|r = H)$. Intuitively, an increase in ϕ can enhance advertising credibility and cause investment to increase by so much that, ceteris paribus, the net gains from false advertising, $\pi^*(q_H^e) - \phi$, actually rise, and prompt a higher y^*. In his seminal discussion, Nelson (1974) suggested that advertising policy may increase the credibility of false advertising. Here, we formalize an even stronger relationship - policy can provide so much credibility that parameters exist where the probability of false advertising is increasing in the level of penalty. Nevertheless, despite any potential increase in y^*, stronger penalties still always induce a larger expected quality, q_H^e. Now consider the optimal penalty:

Proposition 5. *Suppose Condition 1 holds and that $L < \hat{q}_v$. A buyer-orientated policymaker i) always sets $\phi > 0$, and ii) sets $\phi < \phi_0$ such that $y^* > 0$ provided $H > q^*(L)$ and $f(\phi_0)/F(\phi_0)$ is sufficiently small.*

To understand this result, rewrite (8) from earlier using $x \equiv x^*(\phi)$ as

$$E(v) = v^*(L) + \underbrace{(H - L)\frac{v^*(q_H^e) - v^*(L)}{q_H^e - L}}_{\text{Price/Persuasion terms}} \times \underbrace{(1 - x^*(\phi))}_{\text{Investment term}}. \tag{15}$$

The second term captures the trade-off between the price and persuasion effects. As in the main model, Condition 1 ensures that this term is increasing in ϕ if and only if $q_H^e < q^*(L)$. The third term relates to a new 'investment' effect. A high quality product generates more buyer surplus than a low quality product. Therefore ceteris paribus, an increase in ϕ is beneficial since it prompts a higher level of investment. Proposition 5 is then explained as follows. Firstly, unlike in the main model, $\phi = 0$ is never optimal because the firm then never invests and so buyers get only $v^*(L)$. Alternatively, for any $\phi > 0$, the firm invests with positive probability and so from (15) buyer surplus strictly exceeds $v^*(L)$. Secondly though, despite this new investment effect, policy may still refrain from completely eliminating false advertising. In particular, this is the case when $H > q^*(L)$ and $f(\phi_0)/F(\phi_0)$ is relatively small. Intuitively the latter restriction on $f(C)$ implies that starting from strong regulation,

$\phi = \phi_0$, a small decrease in ϕ only has a small effect on the investment probability, such that the combined price and persuasion effects dominate. Consequently, as in the main model, false advertising can sometimes benefit buyers.[16]

Finally, one can compare the optimal penalty with that under exogenous quality. In particular, let ϕ^*_{en} denote the optimal penalty with endogenous quality, and impose a technical condition $f(\phi)(H - L) < 1$ to ensure it is unique. Then, to make a comparison, let $x^*(\phi^*_{en})$ be the proportion of low types under exogenous quality, and denote the associated optimal penalty by ϕ^*_{ex}. One can then prove that $\phi^*_{ex} \leq \phi^*_{en}$, such that the optimal penalty is stronger when quality is endogenous due to the existence of the investment effect.

6 Alternative Equilibria

So far we have studied equilibria in which the firm signals high quality through its report but not its price under Restriction 2. In this section we dispense with this restriction and explore alternative equilibria in which price signaling may occur. Section 6.1 first shows that our equilibrium from Proposition 1 survives common equilibrium refinements and that such refinements have little bite in our game due to the assumption of symmetric costs. Therefore to allow the refinements to have more power, Section 6.2 then introduces a small asymmetry in the types' marginal costs. We show that a Pareto-based criterion uniquely selects a different but related equilibrium where price signaling may arise. Although this equilibrium lacks some attractive features, its implications for policy are qualitatively similar to those already derived.

6.1 Symmetric Costs

Under symmetric costs, we now retain Restriction 1 such that $r(H) = H$ (for the reasons outlined earlier) but relax Restriction 2. To simplify the exposition, we also focus on equilibria in which the firm plays a pure strategy price conditional on a given report. Nevertheless, as is typical in signaling games, there are still a lot of potential equilibria. In particular, there is a continuum of pooling equilibria when $\phi < \phi_1$, a continuum of semi-pooling equilibria when $\phi < \phi_0$, and a continuum of separating equilibria for all $\phi > 0$.[17] Moreover as we now demonstrate, common refinements have little bite due to the absence of single-crossing in price following a high report. For instance, first consider a Pareto-based criterion which rules out a PBE ω if it offers a strictly lower payoff to one sender type and a weakly lower payoff to the other sender type, relative to some other PBE ω'.

[16]Similarly, a welfare-maximizing policymaker may still induce $y^* > 0$.
[17]The interested reader is directed to Section B of the Supplementary Appendix for full details.

Lemma 6. *Given Restriction 1, the Pareto criterion selects the following:*
i) When $\phi < \phi_1$ *the types pool on* $\{r = H, p^*(\bar{q})\}$.
ii) When $\phi \in [\phi_1, \phi_0)$ *the types semi-pool. The low type plays* $\{r = H, p^{sp}\}$ *with any probability* $y \in [0, y^*]$, *and* $\{r = L, p^*(L)\}$ *with probability* $1 - y$. *The high type plays* $\{r = H, p^{sp}\}$, *where* p^{sp} *satisfies*

$$\pi^*(L) = p^{sp}\Big[1 - G\Big(p^{sp} - \frac{xyL + (1-x)H}{1 - x + xy}\Big)\Big] - \phi,$$

and where y^* *is the same as in Proposition 1.*
iii) When $\phi \geq \phi_0$ *the types separate. The low type plays* $\{r = L, p^*(L)\}$ *and the high type plays* $\{r = H, p^*(H)\}$.

The Pareto criterion admits our equilibrium from Proposition 1. However while the Pareto criterion uniquely selects pooling when $\phi < \phi_1$ and separating when $\phi \geq \phi_0$, it is consistent with a broader continuum of semi-pooling equilibria when $\phi \in [\phi_1, \phi_0)$. These semi-pooling equilibria differ in their levels of false advertising, y, and high report price, p^{sp}. Intuitively, for different levels of y, the price p^{sp} adjusts to keep the low type willing to randomize. Further, in each semi-pooling equilibrium, the low type earns $\pi^*(L)$ and the high type earns $\pi^*(L) + \phi$.

Second, consider another common refinement, D1. This has little bite in our game because, as explained above, the types' payoff functions are not single-crossing in price following a high report. In particular, the set of D1-equilibria form a (weak) superset of those consistent with the Pareto criterion. Hence our equilibrium from Proposition 1 also remains consistent with D1.

6.2 Asymmetric Costs

Given such equilibrium multiplicity, the main model used Restriction 2 to uniquely select our chosen equilibrium. However, as an alternative, we now introduce a small asymmetry in the types' marginal costs. This still generates a large number of equilibria but it allows some common refinements to have more power by introducing single-crossing following $r = H$. In particular, now let $c(i)$ denote the marginal cost of type $i = L, H$ and assume $c(L) = 0 < c(H)$. We further denote $\pi(p, q^e; i)$ as the gross profit of a firm with marginal cost $c(i)$ when it charges price p and is expected to have quality q^e. Also let $p^*(q^e; i) = \arg\max_p \pi(p, q^e; i)$ and $\pi^*(q^e; i) = \pi(p^*(q^e; i), q^e; i)$.

We start by considering equilibria of the same economically appealing form as that in Proposition 1. In particular consider a candidate equilibrium in which the high type sets $\{r = H, p^{sp}\}$, whilst the low type randomizes between pooling with probability y, and setting $\{r = L, p^*(L; L)\}$ with probability $1 - y$. Since the types have different costs they now disagree over their preferred pooling price p^{sp}. Suppose they pool on the high type's preferred price $p^*(q^e_H; H)$. This is consistent with the high type having some price 'leadership' since it is the one being mimicked. Provided $c(H) - c(L)$ is not too large, it is then straightforward to prove that there exists an equilibrium which is qualitatively the same as in Proposition 1:

Remark 2. Suppose $c(H) - c(L)$ is not too large. There exists a PBE in which:
i) A high type firm claims $r = H$ and charges $p^*(q^e_H; H)$.
ii) A low type firm randomizes. With probability y^* it claims $r = H$ and charges $p^*(q^e_H; H)$. With probability $1 - y^*$ it claims $r = L$ and charges $p^*(L; L)$.
- When $\phi \leq \phi'_1 \equiv \pi^*(p^*(\bar{q}; H), \bar{q}; L) - \pi^*(L)$, $y^* = 1$.
- When $\phi \geq \phi'_0 = \pi^*(p^*(H; H), H; L) - \pi^*(L)$, $y^* = 0$.
- When $\phi \in (\phi'_1, \phi'_0)$, $y^* \in (0, 1)$ and uniquely solves

$$\pi^*(p^*(q^e_H; H), q^e_H; L) - \phi = \pi^*(L). \tag{16}$$

iii) q^e_H is given by (6). Buyer beliefs are such that $\Pr(q = H | \{r, p\} = \{H, p^*(q^e_H; H)\}) = \frac{1-x}{1-x+xy^*}$ and $\Pr(q = H | \{r, p\} \neq \{H, p^*(q^e_H; H)\}) = 0$.

In particular for intermediate values of ϕ, false advertising still arises in equilibrium, varies smoothly with the policy strength, and induces buyers to positively update their priors. Moreover, after slightly modifying Conditions 1 and 2 to take account of the cost asymmetry it remains true that for relatively high values of L, H, and $(1 - x)$ both a buyer- and a welfare-orientated policymaker would permit a strictly positive level of false advertising.[18]

We now consider other possible forms of equilibria. To ease exposition and to facilitate a comparison with the main model, we henceforth focus on the limit case where $c(H) \to c(L)$. Firstly, under cost asymmetry, it is well-known that the D1 refinement uniquely selects the following least-cost separating equilibrium:

Remark 3. Suppose Restriction 1 holds. In the limit as $c(H) \to c(L)$, the least-cost separating equilibrium has the low type report $r = L$ and charge $p^*(L)$, and the high type report $r = H$ and charge p^s, where $p^s = p^*(H)$ when $\phi \geq \phi_0$, and otherwise p^s is the largest solution to $p[1 - G(p - H)] - \phi = \pi^*(L)$.

Intuitively when ϕ is sufficiently large it is too costly for the low type to mimic. Consequently the high type signals only through its report, and is able to charge the full information price $p^*(H)$. However, for lower ϕ, in addition to using a high report, the high type must also signal by distorting its price above $p^*(H)$ to further deter the low type. Notice however,

[18]However, relative to symmetric costs, there is now an additional reason to eradicate false advertising because a lying low type must distort its price upwards at $p^*(q^e_H; H)$ instead of its preferred (lower) price $p^*(q^e_H; L)$. This distortion provides a further loss to buyer surplus and total welfare.

that in this equilibrium the low type never reports $r = H$. Hence this equilibrium cannot explain the existence of false advertising, and so we do not consider it further. Mailath *et al* (1993) also provide a number of wider arguments against the de facto selection of least-cost separating equilibria. Secondly then, consider the Pareto-based criterion.

Lemma 7. *Suppose Restriction 1 holds. As $c(H) \to c(L)$ the Pareto criterion selects:*
i) If $\phi < \phi_1$, the types pool on $\{r = H, p^*(\bar{q})\}$.
ii) If $\phi \in [\phi_1, \phi_0)$, the types fully separate on $\{r = L, p^*(L)\}$ and $\{r = H, p'\}$ where $p' > p^*(H)$ and is the largest solution to $p[1 - G(p - H)] - \phi = \pi^*(L)$.
iii) If $\phi \geq \phi_0$, the types fully separate on $\{r = L, p^*(L)\}$ and $\{r = H, p^*(H)\}$.

The Pareto criterion now selects a unique equilibrium.[19] As under symmetric costs, it still selects pooling when $\phi < \phi_1$, and separation with full information prices when $\phi \geq \phi_0$. However, when $\phi \in [\phi_1, \phi_0)$, the Pareto criterion now makes a unique prediction. In particular, due to single-crossing, it selects the least-cost separating equilibrium where the high type signals its quality through both its report and its distorted price.

Importantly, this equilibrium still exhibits false advertising for sufficiently low ϕ. However, compared to our equilibrium in Proposition 1 (and its asymmetric cost counterpart above), it lacks two economically attractive features: false advertising i) never influences buyers' beliefs beyond their priors, and ii) does not vary smoothly in ϕ. Nevertheless, despite these differences, we now show that the policy implications are qualitatively similar to those in our original analysis. The following is a useful preliminary result:[20]

Remark 4. The policymaker optimally chooses ϕ such that price signaling does not arise in equilibrium. In particular, any $\phi \in [\phi_1, \phi_0)$ is strictly dominated under a buyer surplus, profit, or total welfare objective.

Intuitively price signaling is costly for both the firm and buyers because price is distorted upwards above the full information level when quality is high. Hence the policymaker prefers to choose either the full information outcome with $\phi \geq \phi_0$ or the full pooling outcome with $\phi < \phi_1$. Equivalently, in terms of our main model, this equates to a choice between $y^* = 0$ and $y^* = 1$ such that the policymaker either fully eliminates or fully permits false advertising. By adapting past results, we can then state the following. Firstly, Proposition 3 implies that a profit-oriented policymaker weakly prefers $y^* = 0$. Secondly, under a buyer surplus objective, the shape of $v^*(q)$ again plays an important role. Under Condition 1 it is immediate that $y^* = 0$ is strictly preferred when $H \leq \hat{q}_v$, and $y^* = 1$ is weakly preferred when $L \geq \hat{q}_v$. For the remaining cases we find that:

[19] This same equilibrium is also uniquely selected by Mezzetti and Tsoulouhas's (2000) Strongly Undefeated Equilibrium refinement.

[20] In the context of a least cost separating equilibrium within a different model, Corts (2013) shows a related result where price signaling is never optimal under a total welfare objective.

Proposition 6. *Suppose that Condition 1 holds and $L < \hat{q}_v < H$:*
i) There exist values of L, H, and $(1-x)$ such that a buyer-oriented policymaker strictly prefers $y^* = 1$ over $y^* = 0$.
ii) The buyer-optimal $y^* \in \{0, 1\}$ is increasing in L, H and $(1-x)$.

Hence, there exists parameters where the price effect dominates the persuasion effect such that buyer surplus is maximized by a positive level of false advertising with $y^* = 1$. Moreover in the same spirit as Corollary 1, $y^* = 1$ is more likely to be optimal when the market is 'healthier'. Under Condition 2, a similar result can also be shown for total welfare. Therefore the policy implications are qualitatively similar to those under our main equilibrium. However these policy implications are somewhat less rich and realistic because, unlike in our main analysis, the optimal level of false advertising is now 'bang-bang' and moderate penalties are never predicted.

Finally, we note that this subsection also contributes to the literatures on quality disclosure and price signaling.[21] First, for a general set of demand conditions, Remark 4 shows that both the firm and buyers always prefer full separation with full information prices or full pooling to the possibility of price signaling. Second, Proposition 6 shows that full pooling can be optimal for buyers and society even when truthful disclosure is costless, and when full separation would involve no output distortions. Third, under our assumptions of two types and a zero cost of truthful disclosure, the monopolist always weakly prefers full separation and so its incentives to fully disclose are weakly excessive relative to the buyer or social optimum.

7 Robustness

This final section shows how the results of the main model are robust to i) an arbitrary number of quality types, and ii) an alternative, multiplicative form of utility.

7.1 An Arbitrary Number of Types

Suppose there are now $n > 2$ quality levels, denoted by $q_1 < ... < q_n$, and that the firm has quality q_i with probability $x_i \in (0, 1)$. To simplify the exposition, let $q_2 > \underset{\sim}{q}$ and $q_{n-1} < \hat{q}_v$ (relaxing these assumptions is straightforward, but adds no new insights). Marginal cost is the same for all types and normalized to zero, while ex ante expected quality is again denoted by $\bar{q} = \sum x_i q_i$. The firm may send any report from the set $Q = \{q_1, ..., q_n\}$, and the policymaker can commit to a richer penalty $\phi(q, r) \geq 0$, which depends on both the firm's actual and reported qualities. We assume that the firm can only be fined if it over-reports its quality i.e. $\phi(q, r) = 0$ for all $r \leq q$. The game and move order are otherwise unchanged.

[21]See Bagwell (2007) and Dranove and Jin (2010) for respective reviews.

As usual, for any particular penalty $\phi(q,r)$ there may exist a large number of PBE. Therefore, for reasons analogous to the main model, we continue to apply versions of Restrictions 1 and 2 in which i) $r(q) \geq q \ \forall q$ such that no type under-reports its quality, and ii) buyer beliefs depend on the firm's claim but not its price. Notice that in any PBE satisfying these restrictions, the penalty function $\phi(q,r)$ induces a mapping from quality types into reports. It is then convenient to let $y^*_{i,j}$ be the probability that a firm of type i claims to have quality j; hence $y^*_{i,i}$ denotes the probability that firm type i sends a truthful report. Letting $\mathbf{y}^*$ be the (triangular) matrix of such probabilities, we may then state:

Lemma 8. *The optimal penalty can be derived in two steps:*
i) First, choose the matrix of probabilities $\mathbf{y}^*$ *which maximizes the policymaker's objective.*
ii) Second, there exists a penalty function $\phi(q,r)$ *which induces the policymaker's optimal* $\mathbf{y}^*$ *as the unique equilibrium outcome of the game.*

Thus conceptually the problem is similar to the two-type case. In particular, analogous to Lemma 2, we can work with the matrix of report probabilities $\mathbf{y}^*$, and be sure that at least one penalty function can implement the desired $\mathbf{y}^*$. Now consider optimal penalties. Given our main model, it is not surprising that under certain conditions the policymaker will permit some false advertising. However once there is an arbitrary number of types, the policymaker also has to decide *which* quality types will be allowed to engage in false advertising, and which quality level(s) they will mimic. To simplify the exposition we now focus on distributions satisfying Conditions 1i and 2i for which $\hat{q}_v = \hat{q}_w = \tilde{q}$:[22]

Proposition 7. *Under Conditions 1i and 2i, the optimal report probabilities are as follows:*
i) Buyer surplus. (a) When $q_n \leq \tilde{q}$ *it is maximized by* $y^*_{i,i} = 1$ *for all* i*. (b) When* $q_n > \tilde{q} > \bar{q}$ *there exists a critical type* i^* *satisfying* $E(q|q \geq q_{i^*}) \leq \tilde{q} < E(q|q \geq q_{i^*+1})$*, such that the optimal solution has* $y^*_{i,i} = 1$ *for all* $i < i^*$*,* $y^*_{i,n} = 1$ *for all* $i > i^*$*, and* $y^*_{i^*,i^*} = 1 - y^*_{i^*,n}$ *where* $y^*_{i^*,n}$ *satisfies:*

$$\frac{x_{i^*} y^*_{i^*,n} q_{i^*} + \sum_{i=i^*+1}^{n} x_i q_i}{x_{i^*} y^*_{i^*,n} + \sum_{i=i^*+1}^{n} x_i} = \tilde{q}.$$

(c) When $\bar{q} \geq \tilde{q}$ *it is maximized by* $y^*_{i,n} = 1$ *for all* i*.*
ii) Profit is maximized by $y^*_{i,i} = 1$ *for all* i*.*
iii) Total welfare. There exists a threshold L^* *such that: (a) When* $q_n \leq \tilde{q}$ *it is maximized by* $y^*_{i,i} = 1$ *for all* i*. (b) When* $q_n > \tilde{q}$ *and* $q_{i^*} \geq L^*$ *it is maximized by the buyer-optimal matrix. (c) When* $q_n > \tilde{q}$ *and* $q_{i^*} < L^*$*, it is maximized by* $y^*_{i,i} = 1$ *for all* i *with* $q_i < L^*$*, and* $y^*_{i,n} = 1$ *for all* i *with* $q_i \geq L^*$*.*

The policymaker induces each firm type to either report truthfully or to claim to have the highest possible quality q_n. Similar to the two-type model, in many cases one firm type is

[22] The optimal pattern of false advertising is qualitatively the same for distributions satisfying the alternative Conditions 1ii and 2ii. Further details are available on request.

required to randomize over its report. Whether buyers gain from false advertising depends upon how the highest quality type q_n compares with $\tilde{q}$. If $q_n \leq \tilde{q}$ the persuasion effect dominates, such that buyers are better off if the firm truthfully reveals its quality. However if $q_n > \tilde{q}$, the highest type has a lot of market power, and so lower types are pooled with it to generate a beneficial price effect. In order to minimize the negative persuasion effect, this pooling is done from the top i.e. first the q_{n-1} type is pooled, then the q_{n-2} type, and so forth, until either $E(q|r = q_n) = \tilde{q}$ or no more types are left to pool. Hence the optimum has full pooling when $\bar{q} > \tilde{q}$, and semi-pooling when $\bar{q} < \tilde{q} < H$. In the latter case, the policymaker permits 'small' lies by types close to q_n, whilst forbidding 'large' lies by types at or close to q_1.

Policy under a total welfare objective also depends upon whether $q_n \gtrless \tilde{q}$. When $q_n \leq \tilde{q}$ a welfare-maximizing policy involves truthful advertising and so coincides with what is optimal for both buyers and the firm. However when $q_n > \tilde{q}$ a welfare-oriented policymaker may allow some lower types to use false advertising in order to raise their output. As with buyer surplus, types with quality closer to q_n are more likely to be allowed to use false advertising since their socially-optimal output levels are highest. Overall, the main insights from the two-type model carry over into this richer multi-type environment.

7.2 Multiplicative Preferences

In the main model a buyer's net utility is additive in quality and equal to $q + \varepsilon - p$. We now show that our results do not qualitatively change when buyers have multiplicative preferences (Mussa and Rosen 1978). In particular, suppose net utility now equals $\theta q - p$, where θ represents a buyer's privately known taste for quality which is drawn independently across buyers using a distribution function $F(\theta)$ with support $[\underline{\theta}, \overline{\theta}]$ where $0 \leq \underline{\theta} < \overline{\theta} \leq \infty$. The associated density $f(\theta)$ is strictly positive, continuously differentiable, and has an increasing hazard rate. The firm has a strictly positive marginal cost denoted by c. All other assumptions remain the same.

First consider the benchmark case in which the firm is known to have quality, q. A buyer purchases the product if and only if $\theta \geq p/q$ such that demand equals $D(p/q) = 1 - F(p/q)$. The firm then chooses its optimal price $p^*(q)$ to maximize $(p - c)\,[1 - F\,(p/q)]$:

Lemma 9. *Suppose the firm is known to have quality q, and define $\underset{\sim}{q}^m = c/\bar{\theta}$. Also define coverage quality $\tilde{q}^m$, where $\tilde{q}^m = cf(\underline{\theta})/\left[\underline{\theta} f\,(\underline{\theta}) - 1\right]$ if $\underline{\theta} f\,(\underline{\theta}) > 1$, and otherwise $\tilde{q}^m = \infty$. Then*

$$p^*(q) = \begin{cases} c & \text{if } q \leq \underset{\sim}{q}^m \\ c + q\frac{1-F(p^*(q)/q)}{f(p^*(q)/q)} & \text{if } q \in \left(\underset{\sim}{q}^m, \tilde{q}^m\right) \\ \underline{\theta} q & \text{if } q \geq \tilde{q}^m \end{cases} \tag{17}$$

When $q \leq \underset{\sim}{q}^m$ quality is too low for the market to be active and so the firm's price is normalized to marginal cost without loss. When instead $q \in \left(\underset{\sim}{q}^m, \tilde{q}^m\right)$ the market is partially covered and $p^*(q)$ satisfies the standard first order condition. Unlike the main model, an increase in q does not cause a parallel upward shift in the firm's demand curve. Nevertheless after differentiating (17) a firm with higher quality still charges more and sells to more buyers. When quality is very large, optimal pricing depends on parameters. Mirroring the main model let $\tilde{q}^m$ denote the quality at which the market becomes covered. Firstly when $\underline{\theta} f\left(\underline{\theta}\right) \leq 1$, $\tilde{q}^m = \infty$ such that the market never becomes fully covered. Intuitively the firm never wishes to sell to buyers with the lowest types because either their willingness-to-pay is too low or there are relatively too few of them. Here $\lim_{q \to \infty} p^*(q)/q = \theta^m$ where $\theta^m > \underline{\theta}$ solves $1 - F(\theta) - \theta f(\theta) = 0$. Secondly when $\underline{\theta} f\left(\underline{\theta}\right) > 1$ we have $\tilde{q}^m < \infty$, such that the market becomes fully covered for sufficiently high quality, after which price tracks the willingness-to-pay of the lowest type, with $p^*(q) = \underline{\theta} q$. As before let $\pi^*(q) = (p^*(q) - c)\left[1 - F\left(p^*(q)/q\right)\right]$ denote the firm's full information profit, which is strictly increasing in $q > \underset{\sim}{q}^m$ and strictly convex in $q \in (\underset{\sim}{q}^m, \tilde{q}^m)$. Moreover for distributions where $\underline{\theta} f\left(\underline{\theta}\right) > 1$, $\pi^*(q)$ is differentiable around $q = \tilde{q}^m$ and then linear in $q > \tilde{q}^m$. We also let $v^*(q) = \int_{p^*(q)}^{\bar{\theta} q} (1 - F(\frac{m}{q}))dm$ denote buyer surplus. As usual the shape of $v^*(q)$ is important for our results, and we return to it below.

Now suppose that the firm is privately informed about its quality. Under Restrictions 1 and 2, one can use similar steps to construct a semi-pooling equilibrium like that in the main model. This has the following implications. Firstly the global convexity of $\pi^*(q)$ implies that ex ante industry profit is again (weakly) maximized by a tough policy which eliminates false advertising. Secondly consider buyer surplus. After some manipulations, expected buyer surplus can again be written as in equation (8) from the main model. It turns out that irrespective of whether $\underline{\theta} f\left(\underline{\theta}\right) \gtreqless 1$, it is possible that a buyer-oriented policymaker permits false advertising. However relative to additive utility the analysis is now more complicated since there are more cases to consider. Therefore in order to simplify the exposition we focus on distributions where $\underline{\theta} f\left(\underline{\theta}\right) > 1$, and impose the following regularity condition:

Condition 1M. There exists a threshold $\hat{q}_v^m = \tilde{q}^m \in (\underset{\sim}{q}^m, \infty)$ such that $v^*(q)$ is strictly convex for $q \in (\underset{\sim}{q}^m, \hat{q}_v^m)$ and linear for $q > \hat{q}_v^m$, with $\lim_{q \uparrow \hat{q}_v^m} v_q^*(q) > \lim_{q \downarrow \hat{q}_v^m} v_q^*(q) > 0$.

Like Condition 1 in the additive utility case, Condition 1M ensures that $v^*(q)$ is s-shaped in q. However notice that unlike in the main model, $v^*(q)$ is strictly increasing in q even once the market is fully covered. Intuitively, under market coverage, price is determined by the willingness-to-pay of the lowest type $\underline{\theta}$, yet higher types benefit more from increases in quality and so collect some rents. Hence Condition 1M ensures that $v^*(q)$ follows an s-shape that is qualitatively more like total welfare within the additive utility case. To understand when Condition 1M may apply, we note the following.

Lemma 10. *Consider a distribution $G(\varepsilon)$ on $[a, b]$ that satisfies Condition 1. Truncate it from below to generate a new distribution $F(\theta)$, which has density $f(\theta) = g(\theta)/[1-G(\underline{\theta})]$ with $\underline{\theta} \geq \max\{a, 0\}$ and $\bar{\theta} = b$. $F(\theta)$ then satisfies Condition 1M with $\hat{q}_v^m = \tilde{q}^m = cf(\underline{\theta})/[\underline{\theta} f(\underline{\theta}) - 1] \in (\underset{\sim}{q}^m, \infty)$ provided i) $\underline{\theta}$ is sufficiently large and ii) $\lim_{\varepsilon \to \underline{\theta}} \sigma(\varepsilon) = -[1-G(\varepsilon)]g'(\varepsilon)/g(\varepsilon)^2 > -\infty$.*

Thus Condition 1M holds for any distribution that satisfies Condition 1, provided it is sufficiently left-truncated. In light of our earlier results, it is then straightforward to verify that i) if $H \leq \hat{q}_v^m$, $y^* = 0$ is optimal; ii) if $L \geq \hat{q}_v^m$ policy is indifferent; and otherwise:

Proposition 8. *Suppose Condition 1M holds and $L < \hat{q}_v^m < H$. Under a buyer surplus objective, there exists a $L^m < \hat{q}_v^m$ such that:*
i) If $L \leq L^m$ the policymaker optimally chooses $y^* = 0$.
ii) If $L > L^m$ the policymaker optimally chooses $y^* \in (0, 1]$.

As in the main model, an increase in false advertising generates related price and persuasion effects, with the former dominating when L and H are relatively large. Finally consider a total welfare objective. Expected total welfare can be expressed as in equation (12) where again $w^*(q) = v^*(q) + \pi^*(q)$. Note under Condition 1M, $w^*(q)$ is guaranteed to be s-shaped as both $v^*(q)$ and $\pi^*(q)$ are strictly convex for $q \in (\underset{\sim}{q}^m, \tilde{q}^m)$ and linear for $q > \tilde{q}^m$. Hence, under Condition 1M, the policy implications are highly related to those in the main model and those under the buyer surplus objective above. Indeed, using familiar methods, it is easy to show that: i) if $H \leq \tilde{q}^m$, $y^* = 0$ is optimal; ii) if $L \geq \tilde{q}^m$ policy is indifferent, but iii) if $L < \tilde{q}^m < H$, a positive level of false advertising, $y^* > 0$, is optimal for sufficiently high L.

8 Conclusions

Despite its prevalence and policy importance, false advertising remains under-studied. This paper presents a model where false claims arise in equilibrium and actively influence the purchase decisions of rational buyers. By utilizing some results on demand curvature, the paper provides precise conditions under which buyers and society benefit from a positive level of false advertising. Intuitively false advertising damages the credibility of high claims, and so counteracts monopoly power via a novel 'price' effect. These results are robust to a variety of extensions, such as endogenous product quality, asymmetric costs, an arbitrary number of quality types, and an alternative multiplicative form of utility.

Our paper is not alone in arguing that policy may wish to induce false advertising by limiting the associated expected penalty. Some other recent papers do this in different settings, where for example firms are uncertain about their quality or compete with an asymmetric rival. The literature on law enforcement also suggests some broader reasons why optimal policy may not be maximal (e.g. Polinsky and Shavell 2007). These include the potential costs of detecting and punishing criminal activity, constraints relating to offenders' ability to pay, and individuals' attempts to avoid prosecution via socially costly activities. Consequently, we do not intend to suggest that our documented price effect is the only reason why it may be optimal to limit advertising regulation. Instead, our paper wishes to highlight that i) the existence of false advertising may benefit buyers and society by reducing the credibility of claims and thereby reducing prices, even in a world where other explanations may not be relevant, ii) such price effects are economically intuitive, robust, and widely applicable across many common market settings, and yet iii) the possibility of such benefits seems to be completely ignored in current advertising regulation. Further theoretical and empirical work to learn more about the optimal level, form, and enforcement of advertising regulation is clearly a promising avenue for future research.

References

[1] Aguirre I., S. Cowan and J. Vickers (2010) "Monopoly Price Discrimination and Demand Curvature" American Economic Review, 100, 1601-1615

[2] Anderson S. and R. Renault (2003) "Efficiency and Surplus Bounds in Cournot Competition" Journal of Economic Theory, 113, 253-264

[3] Anderson S. and R. Renault (2006) "Advertising Content" American Economic Review, 96, 93-113

[4] Armstrong M. and Y. Chen (2017) "Discount Pricing" Working paper

[5] Armstrong M. and J. Zhou (2016) "Search Deterrence" Review of Economic Studies, 83, 26-57

[6] Bagnoli M. and T. Bergstrom (2005) "Log-concave Probability and its Applications" Economic Theory, 26, 445-469

[7] Bagwell K. (2007) "The Economic Analysis of Advertising" in the 'Handbook of Industrial Organization, vol. 3', M. Armstrong and R. Porter (eds.), North-Holland: Amsterdam

[8] Barigozzi F., P.G. Garella, and M. Peitz (2009) "With a Little Help from My Enemy: Comparative Advertising as a Signal of Quality" Journal of Economics and Management Strategy, 18, 1071-1094

[9] Beales H., R. Craswell, and S.C. Salop (1981) "The Efficient Regulation of Consumer Information" The Journal of Law and Economics, 24, 491-539

[10] Bulow J.I. and P. Pfleiderer (1983) "A Note on the Effect of Cost Changes on Prices" Journal of Political Economy, 91, 182-185

[11] Cawley J., M. Eisenberg and R. Avery (2013) "The Effect of Deceptive Advertising on Consumption of the Advertised Good and its Substitutes: The Case of Over-the-Counter Weight Loss Products" Working paper

[12] Chakraborty A. and R. Harbaugh (2014) "Persuasive Puffery" Marketing Science, 33, 382-400

[13] Chen Y. and M. Schwartz (2015) "Differential Pricing When Costs Differ: A Welfare Analysis" Rand Journal of Economics, 46, 442-460

[14] Corts K. (2013) "Prohibitions on False and Unsubstantiated Claims: Inducing the Acquisition and Revelation of Information Through Competition Policy" Journal of Law and Economics, 56, 453-486

[15] Corts K. (2014a) "The Social Value of Information on Product Quality" Economics Letters, 122, 140-143

[16] Corts K. (2014b) "Finite Optimal Penalties for False Advertising" Journal of Industrial Economics, 62, 661-681

[17] Cowan S. (2012) "Third-Degree Price Discrimination and Consumer Surplus" Journal of Industrial Economics, 60, 333-345

[18] Darke P.R. and R.J.B. Ritchie (2007) "The Defensive Consumer: Advertising, Deception, Defensive Processing, and Distrust" Journal of Marketing Research, 44, 114-127

[19] Daughety A.F. and J.F. Reinganum (1997) "Everybody Out of the Pool: Products Liability, Punitive Damages, and Competition" Journal of Law, Economics, and Organization, 13, 410-432

[20] Dranove D. and G. Jin (2010) "Quality Disclosure and Certification: Theory and Practice" Journal of Economic Literature, 48, 935-963

[21] Drugov M. and M. Troya-Martinez (2015) "Vague Lies and Lax Standards of Proof: On the Law and Economics of Advice" Working paper

[22] Glaeser E. and G. Ujhelyi (2010) "Regulating Misinformation" Journal of Public Economics, 94, 247-257

[23] Hattori K. and K. Higashida (2012) "Misleading Advertising in Duopoly" Canadian Journal of Economics, 45, 1154-1187

[24] Inderst R. and M. Ottaviani (2013) "Sales Talk, Cancellation Terms, and the Role of Consumer Protection" Review of Economic Studies, 80, 1002-1026

[25] Johnson J.P and D.P. Myatt (2006) "On the Simple Economics of Advertising, Marketing, and Product Design" American Economic Review, 96, 756-784

[26] Kamenica E. and M. Gentzkow (2011) "Bayesian Persuasion" American Economic Review, 101, 2590-2615

[27] Kartik N. (2009) "Strategic Communication with Lying Costs" Review of Economic Studies, 76, 1359-1395

[28] Mailath G.J., M. Okuno-Fujiwara and A. Postlewaite (1993) "Belief-Based Refinements in Signaling Games" Journal of Economic Theory, 60, 241-276

[29] Mayzlin D., Y. Dover and J. Chevalier (2014) "Promotional Reviews: An Empirical Investigation of Online Review Manipulation" American Economic Review, 104, 2421-55

[30] Mezzetti C. and T. Tsoulouhas (2000) "Gathering Information Before Signing a Contract with a Privately Informed Principal" International Journal of Industrial Organization, 18, 667-689

[31] Mussa M. and S. Rosen (1978) "Monopoly and Product Quality" Journal of Economic Theory, 18, 301-317

[32] Nelson P. (1974) "Advertising as Information" Journal of Political Economy, 82, 729-754

[33] Newell S.J., R.E. Goldsmith, and E.J. Banzhaf (1998) "The Effect of Misleading Environmental Claims on Consumer Perceptions of Advertisements" Journal of Marketing Theory and Practice, 6, 48-60

[34] Piccolo S., P. Tedeschi and G. Ursino (2015) "How Limiting Deceptive Practices Harms Consumers" Rand Journal of Economics, 46, 611-624

[35] Polinsky A. M. and S. Shavell (2007) "The Theory of Public Enforcement of Law" in the 'Handbook of Law and Economics, vol.1', A.M. Polinsky and S. Shavell (eds.), North-Holland, Amsterdam

[36] Rao A. and E. Wang (2016) "Demand for "Healthy" Products: False Claims and FTC Regulation" Working paper

[37] Renault R. (2015) "Advertising in Markets" in the 'Handbook of Media Economics', S.P. Anderson, D. Stromberg, and J. Waldfogel (eds.), Elsevier

[38] Rhodes A. and C.M. Wilson (2017) "False Advertising" Working paper

[39] Weyl G. and M. Fabinger (2013) "Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition" Journal of Political Economy, 121, 528-583

[40] Zinman J. and E. Zitzewitz (2016) "Wintertime for Deceptive Advertising?" American Economic Journal: Applied Economics, 8, 177-192

Appendix

Proof of Lemma 1. i) If $q \leq \underset{\sim}{q}$ demand is zero for all $p \geq 0$, so profit is weakly maximized at $p^* = 0$. ii) If $q > \underset{\sim}{q}$ profit is strictly increasing in $p < a + q$, therefore the optimal price must satisfy $p^* \geq a + q$. At an interior solution, the first order condition is

$$1 - pg\,(p - q) / \left[1 - G\,(p - q)\right] = 0. \tag{18}$$

a) When $q \in \left(\underset{\sim}{q}, \tilde{q}\right)$ the left-hand side of (18) is strictly positive as $p \to a + q$, strictly negative as $p \to b + q$, and strictly decreasing in p because $1 - G$ is logconcave. Hence a unique p^* solves equation (18). Define $\sigma(\psi) = -[1 - G(\psi)]g'(\psi)/g(\psi)^2$. Differentiating (18) gives $p_q^*(q) = (1 - \sigma(p^*(q) - q))/(2 - \sigma(p^*(q) - q))$, which lies in $[0, 1)$ because logconcavity of $1 - G$ implies $\sigma(\psi) \leq 1$. b). When $q \geq \tilde{q}$ the left-hand side of (18) is strictly negative at all $p > a + q$ and hence $p^* = a + q$. □

Proof of Proposition 1. a) Using Restriction 2 we can define $\beta_H^e = \Pr\left(q = H | r = H\right)$ and $q_H^e = \left(1 - \beta_H^e\right) L + \beta_H^e H$. Conditional on buyer beliefs, after reporting $r = H$ the firm's price must maximize its profit and so it charges $p^*(q_H^e)$. b) Using Restriction 1, Bayes' rule implies that $\Pr(q = H | r = L, p\} = 0$ if $\{r = L, p\}$ is played in equilibrium. Therefore if in equilibrium the firm reports $r = L$, it must also charge $p^*(L)$. c) As $y^* = \Pr\left(r(L) = H\right)$, Bayes' rule implies $\beta_H^e = (1 - x)/(1 - x + xy^*)$. However Bayes' rule places no restriction on beliefs following a $\{r = L, p\}$ which is off-path. d) y^* must be consistent with the low type behaving optimally. Firstly given $y^* = 0$, $r(L) = L$ is weakly dominant if and only if $\phi \geq \phi_0$. Secondly given $y^* = 1$, $r(L) = H$ is weakly dominant if and only if $\phi \leq \pi^*(\bar{q}) - \pi^*\left(q_L^e\right)$ where $\pi^*\left(q_L^e\right)$ is the maximum profit (given buyer beliefs) attainable with report $r = L$. This condition holds for any $\phi \leq \phi_1$ given appropriate off-path beliefs, such as those specified in the proposition. Thirdly given $y^* \in (0, 1)$, the low type must be indifferent between $r = L$ and $r = H$ i.e. (5) must hold. Moreover $y^* \in (0, 1)$ implies $q_H^e \in (\bar{q}, H)$, such that equation

(5) cannot hold for $\phi \notin (\phi_1, \phi_0)$, but has a unique solution for any $\phi \in (\phi_1, \phi_0)$. e) Finally, given buyer beliefs, it is indeed optimal for the high type to report $r(H) = H$. $\square$

Proof of Lemma 3. i) Consider $L < \hat{q}_v$. a) Under Condition 1i the shape of $v^*(q)$ implies $\hat{q}_v = \tilde{q} < \infty$ and also that (10) is strictly negative for $q_H^e < \tilde{q}$ and strictly positive for $q_H^e > \tilde{q}$. Hence $q^*(L) = \tilde{q}$. b) Under Condition 1ii the shape of $v^*(q)$ implies that $\tilde{q} = \infty$, and hence $v^*(q)$ is continuously differentiable on $(\underset{\sim}{q}, \tilde{q})$. This all implies that (10) is strictly negative for $q_H^e < \hat{q}_v$, continuous at $q_H^e = \hat{q}_v$, strictly increasing in q_H^e for $q_H^e > \hat{q}_v$, and strictly positive for sufficiently high q_H^e. To see the latter, note that we can write for $q_H^e > \hat{q}_v$ that

$$\frac{\partial E(v)}{\partial y^*} \propto \int_{\hat{q}_v}^{q_H^e} \left[v_q^*(z) - v_q^*(q_H^e)\right] \mathrm{d}z + \int_{\max\{\underset{\sim}{q},L\}}^{\hat{q}_v} \left[v_q^*(z) - v_q^*(q_H^e)\right] \mathrm{d}z - (\max\{\underset{\sim}{q}, L\} - L)v_q^*(q_H^e)\,, \tag{19}$$

which is strictly positive as $q_H^e \to \infty$ because $v_q^*(z) > v_q^*(q_H^e)$ for all $z \in (\hat{q}_v, q_H^e)$, and because $\lim_{q\to\infty} v_q^*(q) = 0$. Consequently (10) has a unique root $q^*(L) > \hat{q}_v$, and is strictly negative for $q_H^e < q^*(L)$ and strictly positive for $q_H^e > q^*(L)$. Furthermore notice that $v_q^*(q^*(L)) > v_q^*(L)$, such that $q^*(L)$ is strictly decreasing in L; also by definition $\lim_{L\to\hat{q}_v} q^*(L) = \hat{q}_v$. c) Therefore $E(v)$ is quasiconcave in ϕ under Condition 1 because q_H^e strictly decreases in y^*, and y^* strictly decreases in $\phi \in [\phi_1, \phi_0]$. ii) Consider $L > \hat{q}_v$. Under Condition 1, $v^*(q)$ is weakly increasing and concave in $q > \hat{q}_v$, so (10) is weakly positive. Hence $E(v)$ is weakly decreasing in ϕ. $\square$

Proof of Proposition 2. i) Note that $q_H^e \leq q^*(L)$ for all ϕ, so by Lemma 3 $E(v)$ is maximized at $\phi^* \geq \phi_0$. ii) Note that $q_H^e < q^*(L)$ when $\phi < \pi^*(q^*(L)) - \pi^*(L)$, and $q_H^e > q^*(L)$ when $\phi > \pi^*(q^*(L)) - \pi^*(L)$. Hence from Lemma, 3 $E(v)$ is maximized at $\phi^* = \pi^*(q^*(L)) - \pi^*(L)$ such that $q_H^e = q^*(L)$. iii) Note that $q_H^e \geq q^*(L)$ for all ϕ, hence by Lemma 3 $E(v)$ is maximized at $\phi^* \leq \phi_1$. Finally, Proposition 1 gives the associated optimal y^* for each case. $\square$

Proof of Corollary 1. Using Proposition 2 optimal false advertising is

$$y^* = \min\left\{\max\left\{\frac{(H - q^*(L))(1-x)}{(H - q^*(L))(1-x) + q^*(L) - \bar{q}}, 0\right\}, 1\right\}. \tag{20}$$

Recall from the proof of Lemma 3 that $q^*(L)$ is weakly decreasing in L. Hence (20) is weakly increasing in L, H, and $(1-x)$. $\square$

Proof of Proposition 3. Given $E(\Pi) = xE(\pi_L) + (1-x)E(\pi_H)$, it is immediate from (11) that a) $E(\Pi) = \pi^*(\bar{q}) - x\phi$ when $\phi < \phi_1$, b) $E(\Pi) = \pi^*(L) + (1-x)\phi$ when $\phi \in [\phi_1, \phi_0]$, and c) $E(\Pi) = x\pi^*(L) + (1-x)\pi^*(H)$ when $\phi > \phi_0$. Hence $E(\Pi)$ is quasiconvex, minimized at ϕ_1, and cannot be maximized at any $\phi \in (0, \phi_0)$. Then for part i), $\phi = \phi_0$ strictly dominates $\phi = 0$ because $\pi^*(q)$ is convex everywhere and strictly convex for $q \in \left(\underset{\sim}{q}, \tilde{q}\right)$. For part ii) note that $\pi^*(q) = a + q$ for all $q \geq \tilde{q}$, and hence $E(\Pi) = a + \bar{q}$ for any $\phi \in \{0\} \cup [\phi_0, \infty)$. $\square$

Proof of Lemma 4. Recall that y^* strictly decreases in $\phi \in [\phi_1, \phi_0]$. i) When $q_H^e < \hat{q}_w$ Condition 2 implies that (14) is strictly negative. ii) Consider $L < \hat{q}_w < q_H^e$. Condition 2 implies that $\lim_{q \uparrow \hat{q}_w} w_q^*(q) > w_q^*(q_H^e) > 0$. (This is clear for Condition 2i. Condition 2ii implies that $\tilde{q} = \infty$ such that $w^*(q)$ is continuously differentiable with $\lim_{q \uparrow \hat{q}_w} w_q^*(q) = w^*(\hat{q}_w)$.) Hence the equation $w_q^*(\check{L}) = w_q^*(q_H^e)$ has a unique solution $\check{L} < \hat{q}_w$. Condition 2 also implies that (14) is strictly increasing in $L < \check{L}$, strictly decreasing in $L > \check{L}$, strictly positive at $L = \check{L}$, and weakly positive as $L \to \hat{q}_w$. In addition because Condition 2 implies $w_q^*(q_H^e) > 0$, (14) is strictly negative for $L < q_H^e - w^*(q_H^e)/w_q^*(q_H^e)$. All of this allows us to conclude that (14) has a unique root $L^*(q_H^e) < \check{L}$, and is strictly negative (positive) for L below (above) $L^*(q_H^e)$. Note that under Condition 2i $\tilde{q} = \hat{q}_w$ such that $L^*(q_H^e)$ is a constant which solves $v^*(\tilde{q}) - v^*(L) + a - \pi^*(L) + L = 0$. Note that under Condition 2ii $L^*(q_H^e)$ is strictly decreasing in q_H^e because (14) is strictly increasing in L evaluated around the point $L = L^*(q_H^e)$, and also in q_H^e. Since in both cases $L^*(q_H^e)$ is weakly decreasing in q_H^e, and q_H^e is increasing in ϕ, $E(w)$ is quasiconcave in ϕ. iii) When $L > \hat{q}_w$ Condition 2 implies that (14) is weakly positive. $\square$

Proof of Proposition 4. First consider $L \leq L^*(H)$. Whenever $q_H^e < \hat{q}_w$ Lemma 4i implies that $E(w)$ is strictly decreasing in y^*. Whenever $q_H^e > \hat{q}_w$ Lemma 4ii implies that $E(w)$ is weakly decreasing in y^* due to the fact that $L^*(q)$ is a weakly decreasing function. Therefore given our tie-break rule the optimum has $y^* = 0$. Second consider $L > L^*(H)$. Lemma 4ii implies that $E(w)$ is strictly increasing in y^* starting from $y^* = 0$, and hence the optimum has $y^* > 0$. To prove that $y^* = 1$ can be optimal, it suffices to consider an example in which $L^*(\bar{q}) < L < \hat{q}_w < \bar{q}$ such that $E(w)$ is strictly increasing everywhere in y^*. $\square$

Proof of Lemma 5. Part i) follows from arguments in the text. For part ii) look for an equilibrium in which a positive measure of types invest. Since $\pi^*(H) - \pi^*(L) < \infty$ not all types invest, hence in any putative equilibrium $x^*(\phi) \in (0, 1)$. Under Restrictions 1 and 2 the equilibrium is semi-pooling with the same form as Proposition 1. a) Consider $\phi = \phi_0$. There is clearly an equilibrium with $y^* = 0$; since a high quality firm earns $\pi^*(H) - \pi^*(L)$ more than a low quality firm, $x^*(\phi_0) = 1 - F(\phi_0)$. There is no equilibrium with $y^* > 0$ as $\pi^*(q_H^e) - \phi_0 < \pi^*(H) - \phi_0 = \pi^*(L)$ such that no firm with $q = L$ would want to report $r = H$. b) Consider $\phi \in (0, \phi_0)$. There is no equilibrium with $y^* = 0$, since $\pi^*(q_H^e) - \phi = \pi^*(H) - \phi > \pi^*(L)$, such that a firm with $q = L$ would deviate and report $r = H$. Therefore look for an equilibrium with $y^* > 0$. A high quality firm earns ϕ more than a low quality firm, hence $x^*(\phi) = 1 - F(\phi)$. Also the gain to a firm with $q = L$ from reporting $r = H$ instead of $r = L$ is $\pi^*(q_H^e) - \phi - \pi^*(L)$, or

$$\pi^* \left(L + (H - L) \frac{F(\phi)}{F(\phi) + y^* (1 - F(\phi))} \right) - \phi - \pi^*(L). \tag{21}$$

This is continuous and strictly decreasing in y^*, and is strictly positive at $y^* = 0$. If (21) is weakly positive at $y^* = 1$ it is strictly positive at all $y^* \in [0, 1)$, hence there is a unique equilibrium with $y^* = 1$. If (21) is strictly negative at $y^* = 1$, there exists a unique equilibrium with $y^* \in (0, 1)$ which makes (21) equal to zero. □

Proof of Proposition 5. i) The proof that $\phi = 0$ is never optimal is given in the text after the proposition. ii) It is enough to show that $\partial E(v)/\partial\phi|_{\phi=\phi_0} < 0$. Note that for $\phi \in (0, \phi_0]$,

$$\pi^* \left(q_H^e(\phi)\right) = \max \left\{\pi^*(L) + \phi, \pi^* \left(L + (H - L)F(\phi)\right)\right\}, \tag{22}$$

where the first part applies when $y^* \in (0, 1)$, and the second part applies when $y^* = 1$. Equation (22) implies that for some small $\delta > 0$, $\pi^* \left(q_H^e(\phi)\right) = \pi^*(L) + \phi$ for all $\phi \in [\phi_0 - \delta, \phi_0]$. Using $dq_H^e/d\phi = 1/\left(d\pi^*(q_H^e)/dq\right)$ and equation (15), $\partial E(v)/\partial\phi|_{\phi=\phi_0}$ is proportional to

$$\frac{(H - L)v_q^*(H) - (v^*(H) - v^*(L))}{\pi_q^*(H)\left(v^*(H) - v^*(L)\right)(H - L)} + \frac{f(\phi_0)}{F(\phi_0)}.$$

The first term is strictly negative since $H > q^*(L)$, and dominates the second term provided $f(\phi_0)/F(\phi_0)$ is sufficiently small. □

All remaining proofs for the paper are in Section C of the Supplementary Appendix.

Supplementary Appendix

Section A: Further Information on Conditions 1 and 2

Lemma 11. *Conditions 1' and 2' below are equivalent to Conditions 1 and 2 respectively:*

Condition 1'. *Let* $z_v(\psi) = -\sigma'(\psi) + [2-\sigma(\psi)]g(\psi)/[1-G(\psi)]$. *The demand function satisfies either i)* $\tilde{q} < \infty$ *and* $z_v(\psi) > 0$ *for all* $\psi \in (a,b)$, *or ii)* $\tilde{q} = \infty$, $z_v(\psi)$ *changes from negative to positive at exactly one value of* $\psi \in (a,b)$, *and* $\lim_{\psi \to a} \sigma(\psi) = -\infty$.

Condition 2'. *Let* $z_w(\psi) = -\sigma'(\psi) + [2-\sigma(\psi)][3-\sigma(\psi)]g(\psi)/[1-G(\psi)]$. *The demand function satisfies either i)* $\tilde{q} < \infty$ *and* $z_w(\psi) > 0$ *for all* $\psi \in (a,b)$, *or ii)* $\tilde{q} = \infty$, $z_w(\psi)$ *changes from negative to positive at exactly one value of* $\psi \in (a,b)$, *and* $\lim_{\psi \to a} \sigma(\psi) = -\infty$.

Proof. We proceed in a number of steps.

Step 1. Simple calculations reveal that $v^*_{qq}(q) \propto z_v(p^*(q) - q)$ and $w^*_{qq}(q) \propto z_w(p^*(q) - q)$ for all $q \in (\underset{\sim}{q}, \tilde{q})$.

Step 2. $p^*(q) - q$ is continuous and strictly decreasing in $q \in (\underset{\sim}{q}, \tilde{q})$, with $\lim_{q \to \underset{\sim}{q}} p^*(q) - q = b$, and also $\lim_{q \to \tilde{q}} p^*(q) - q = a$. The first part is straightforward and follows from arguments in the text. The second part follows from the definition $\underset{\sim}{q} = -b$, and the fact that $\lim_{q \to \underset{\sim}{q}} p^*(q) = 0$. The third part is immediate when $\tilde{q} < \infty$; when $\tilde{q} = \infty$, if to the contrary we have $\lim_{q \to \infty} p^*(q) - q = \psi' > a$ then we also have $\lim_{q \to \infty} p^*(q) = \infty$, but this violates the interior first order condition (18).

Step 3. Condition 1i implies 1i', and Condition 2i implies 2i'. (A proof of the converse is straightforward and omitted.) a) Since $v^*(q)$ and $w^*(q)$ are not differentiable around $\hat{q}_v$ and $\hat{q}_w$ respectively, we must have $\hat{q}_v = \hat{q}_w = \tilde{q} < \infty$. b) Given Step 2, $v^*_{qq}(q) > 0$ for all $q \in (\underset{\sim}{q}, \tilde{q})$ implies that $z_v(\psi) > 0$ for all $\psi \in (a,b)$. Similarly $w^*_{qq}(q) > 0$ for all $q \in (\underset{\sim}{q}, \tilde{q})$ implies that $z_w(\psi) > 0$ for all $\psi \in (a,b)$.

Step 4. Condition 1ii implies 1ii', and Condition 2ii implies 2ii'. (A proof of the converse is straightforward and omitted.) a) Since $v^*(q)$ and $w^*(q)$ are strictly increasing for all $q > \underset{\sim}{q}$ we must have $\tilde{q} = \infty$. b) Since $v^*(q)$ and $w^*(q)$ are continuously differentiable, $v^*_{qq}(q)$ and $w^*_{qq}(q)$ are continuous. Combined with Step 2, $v^*_{qq}(q) \gtreqless 0$ for $q \lesseqgtr \hat{q}_v$ implies that $z_v(\psi)$ changes from negative to positive at exactly one value of $\psi \in (a,b)$; similarly $w^*_{qq}(q) \gtreqless 0$ for $q \lesseqgtr \hat{q}_w$ implies that $z_w(\psi)$ changes from negative to positive at exactly one value of $\psi \in (a,b)$. c) Notice that for $q > \underset{\sim}{q}$ we have $v^*_q(q) = \frac{1-G(p^*(q)-q)}{2-\sigma(p^*(q)-q)}$ and $w^*_q(q) = \frac{[3-\sigma(p^*(q)-q)][1-G(p^*(q)-q)]}{2-\sigma(p^*(q)-q)}$. Using Step 2 $\lim_{q \to \infty} v^*_q(q) = 0$ and $\lim_{q \to \infty} w^*_q(q) = 1$ both imply $\lim_{\psi \to a} \sigma(\psi) = -\infty$. $\square$

Now consider the following generalized setting in which demand equals $s\left[1-G\left(\frac{p-q-\mu}{m}\right)\right]$, where μ is a location parameter and $m, s \in (0,\infty)$ are stretch parameters (Weyl and Tirole 2012). This corresponds to a setting in which a mass $s>0$ of buyers have unit demand, and each buyer's valuation is given by $q+\mu+m\varepsilon$ with ε distributed according to $G(\varepsilon)$. In the main text we focus on the case $\mu=0$ and $m=s=1$. However in fact:

Claim 1. If Conditions 1 and 2 hold for a demand $1-G(p-q)$, they also hold for any generalized demand of the form $s\left[1-G\left(\frac{p-q-\mu}{m}\right)\right]$.

Proof. Consider Condition 1. In light of Lemma 11 it is sufficient to check that Condition 1' holds. The market coverage point for this generalized demand is $\tilde{q}(s,m,\mu)=-\mu+m\left(-a+1/g(a)\right)$, hence $\tilde{q}(s,m,\mu)<\infty$ if and only if $\tilde{q}<\infty$. Let $\sigma(\psi;s,m,\mu)$ be the curvature of the generalized demand form. We may then write the analogue of $z_v(\psi)$ for this new demand as

$$z_v(\psi;s,m,\mu)=-\frac{d\sigma(\psi;s,m,\mu)}{d\psi}+\left[2-\sigma(\psi;s,m,\mu)\right]\left[\frac{dsG\left(\frac{\psi-\mu}{m}\right)}{d\psi}\Big/ s\left[1-G\left(\frac{\psi-\mu}{m}\right)\right]\right].$$

After solving for $\sigma(\psi;s,m,\mu)$ and substituting it in, then canceling terms:

$$z_v(\psi;s,m,\mu)=\frac{1}{m}\left[-\sigma'\left(\frac{\psi-\mu}{m}\right)+\left[2-\sigma\left(\frac{\psi-\mu}{m}\right)\right]\frac{g\left(\frac{\psi-\mu}{m}\right)}{1-G\left(\frac{\psi-\mu}{m}\right)}\right]\propto z_v\left(\frac{\psi-\mu}{m}\right).$$

Moreover by the same logic as above $\lim_{q\to\tilde{q}(s,m,\mu)}\psi=a$, and simple calculations reveal that in the case of Condition 1ii $\lim_{\psi\to a}\sigma(\psi)=-\infty$ if and only if $\lim_{\psi\to a}\sigma(\psi;s,m,\mu)=-\infty$. Hence $z_v(\psi;s,m,\mu)$ satisfies Condition 1' if and only if $z_v(\psi)$ satisfies it. The proof for Condition 2 is very similar and so is omitted. □

Specific Examples

We now show that Conditions 1 and 2 are satisfied by a wide range of common demand curves. In light of Lemma 11 and Claim 1, it is sufficient to check Conditions 1' and 2' for the case where $s=m=1$ and $\mu=0$. For further related background material see Appendix E in Fabinger and Weyl (2016).

1. Generalized Pareto Distribution: $G(\psi)=1-\left(1-\frac{(1-\sigma)\psi}{(2-\sigma)}\right)^{\frac{1}{1-\sigma}}$ on $\left[0,\frac{2-\sigma}{1-\sigma}\right)$ for $\sigma<1$, and $G(\psi)=1-e^{-\psi}$ on $[0,\infty)$ for $\sigma=1$. Special cases include the Uniform ($\sigma=0$) and Exponential ($\sigma=1$) distributions. Note that $\tilde{q}=(2-\sigma)<\infty$ and $\sigma(\psi)=\sigma$. Hence Conditions 1i' and 2i' are satisfied, because $z_v(\psi)\propto(2-\sigma)>0$ and $z_w(\psi)\propto(3-\sigma)(2-\sigma)>0$.

2. Normal: $G(\psi) = \int_{-\infty}^{\psi} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$ on $(-\infty, \infty)$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, and $\sigma(\psi) = \frac{\psi[1-G(\psi)]}{g(\psi)}$ because $g'(\psi) = -\psi g(\psi)$. Hence $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Moreover

$$z_v(\psi) \propto 2 \left(\frac{g(\psi)}{1 - G(\psi)} \right)^2 - 1 - \psi^2. \tag{23}$$

Condition 1ii' is satisfied because (23) is negative as $\psi \to -\infty$, is strictly increasing in $\psi \leq 0$ since $\frac{g(\psi)}{1-G(\psi)}$ is strictly increasing, and is strictly positive for all $\psi \geq 0$. To prove the latter, note that for all $\psi \geq 0$ we have the lower bound $\frac{g(\psi)}{1-G(\psi)} \geq \frac{\psi + \sqrt{\psi^2 + 8/\pi}}{2}$ (see Duembgen 2010). In addition

$$z_w(\psi) \propto 6 \left(\frac{g(\psi)}{1 - G(\psi)} \right)^2 - 4\psi \frac{g(\psi)}{1 - G(\psi)} - 1 = 6 \left(\frac{g(\psi)}{1 - G(\psi)} \right)^2 + 4 \frac{g'(\psi)}{1 - G(\psi)} - 1. \tag{24}$$

Condition 2ii' is satisfied. Firstly as $\psi \to -\infty$, (24) tends to -1. Secondly (24) is strictly increasing in $\psi < -1$, because $\frac{g(\psi)}{1-G(\psi)}$ and $g'(\psi) > 0$ are both strictly increasing. Thirdly (24) is strictly positive for all $\psi \in [-1, 0]$. This can be proved by noting that on this interval, we have the lower bound $g(\psi) \geq \left(1 - \frac{\psi^2}{2}\right) / \sqrt{2\pi}$, and the upper bound $1 - G(\psi) \leq \frac{1}{2} - xg(0)$. Fourthly (24) is also strictly positive for all $\psi > 0$. This can be proved by noting that $\frac{g(\psi)}{1-G(\psi)}$ strictly increasing implies both $2\left(\frac{g(\psi)}{1-G(\psi)}\right)^2 > 2\left(\frac{g(0)}{1-G(0)}\right)^2 > 1$ and $4\left[\left(\frac{g(\psi)}{1-G(\psi)}\right)^2 + \frac{g'(\psi)}{1-G(\psi)}\right] > 0$.

3. Weibull: $G(\psi) = 1 - e^{-\psi^\alpha}$ on $[0, \infty)$ where $\alpha > 1$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = 1 - \left(\frac{\alpha - 1}{\alpha \psi^\alpha}\right)$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Moreover

$$z_v(\psi) \propto (\alpha - 1)(\psi^\alpha - 1) + \alpha \psi^{2\alpha} \text{ and } z_w(\psi) \propto 2\alpha^2 \psi^{2\alpha} + 3\alpha(\alpha - 1)\psi^\alpha - (\alpha - 1) \tag{25}$$

Conditions 1ii' and 2ii' are both satisfied, since both expressions in (25) are strictly negative as $\psi \to 0$, strictly increasing in ψ and strictly positive as $\psi \to \infty$.

4. Power: $G(\psi) = \psi^c$ on $(0, 1]$ where $c > 1$. Note that $\underset{\sim}{q} = -1$, $\tilde{q} = \infty$, $\sigma(\psi) = \frac{c-1}{c}[1 - \psi^{-c}]$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Moreover

$$z_v(\psi) \propto \frac{(c+1)}{(c-1)}(\psi^c)^2 + 2\psi^c - 1 \text{ and } z_w(\psi) \propto (\psi^c)^2 \Big[\frac{(2c+1)(c+1)}{c-1}\Big] + (4c+2)\psi^c - 1 \tag{26}$$

Conditions 1ii' and 2ii' are both satisfied, since both expressions in (26) are strictly negative as $\psi \to 0$, strictly increasing in ψ and strictly positive as $\psi \to 1$.

5. Type I Extreme Value (Max version): $G(\psi) = e^{-e^{-\psi}}$ on $(-\infty, \infty)$. Note $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = (e^\psi - 1)(e^{e^{-\psi}} - 1)$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Numerical simulations show that Conditions 1ii' and 2ii' are both satisfied.

6. Logistic: $G(\psi) = \frac{e^{\psi}}{1+e^{\psi}}$ on $(-\infty, \infty)$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = 1 - e^{-\psi}$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Condition 1ii' is satisfied because $z_v(\psi) \propto e^{2\psi} - 1$, which is single-crossing from negative to positive at $\psi = 0$. However Condition 2' is not satisfied since $z_w(\psi) \propto 2 + 2e^{-\psi}$, which is strictly positive everywhere.

7. Type I Extreme Value (Min version): $G(\psi) = 1 - e^{-e^{\psi}}$ on $(-\infty, \infty)$. Note that $\underset{\sim}{q} = -\infty$, $\tilde{q} = \infty$, $\sigma(\psi) = 1 - e^{-\psi}$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Condition 1ii' is satisfied because $z_v(\psi) \propto e^{-\psi}(1 - e^{-\psi}) + 1$, which is single-crossing from negative to positive at $\psi = \ln\left(\frac{-1+\sqrt{5}}{2}\right)$. However Condition 2' is not satisfied since $z_w(\psi) \propto 2 + 3e^{-\psi}$, which is strictly positive everywhere.

8. Mirror Image Pareto: $G(\psi) = (-\psi)^{-\beta}$ on $(-\infty, -1]$ with $\beta > 0$. Note that $\underset{\sim}{q} = 1$, $\tilde{q} = \infty$, $\sigma(\psi) = \frac{[1-(-\psi)^{\beta}](1+\beta)}{\beta}$ and $\lim_{\psi \to a} \sigma(\psi) = -\infty$. Define $h = (-\psi)^{\beta}$. Condition 1ii' is satisfied because $z_v(\psi) \propto -h^2 + 2h + \frac{\beta-1}{\beta+1}$, which is single-crossing from negative to positive as it is strictly negative as $\psi \to -\infty$, strictly increasing in ψ, and strictly positive as $\psi \to -1$. However Condition 2' is not satisfied as $z_w(\psi) \propto h^2 + 2(2\beta - 1)h + \frac{(\beta-1)(2\beta-1)}{(1+\beta)}$, which is strictly positive everywhere.

Section B: Further Information on Alternative Equilibria

This section provides further information on the alternative equilibria discussed in Section 6.1 under symmetric costs.

Pooling equilibria: Here both types report $r = H$ and charge a price p^p which satisfies

$$\pi^*(L) \leq p^p[1 - G(p^p - \bar{q})] - \phi \,. \tag{27}$$

To understand this inequality, note that on the equilibrium path buyers maintain their prior belief that quality is $\bar{q}$. Therefore in a pooling equilibrium the two types earn $p^p[1 - G(p^p - \bar{q})] - \phi$ and $p^p[1 - G(p^p - \bar{q})]$ respectively. Even with the most pessimistic off-path beliefs, each type can guarantee a payoff $\pi^*(L)$ by reporting $r = L$ and charging $p^*(L)$. Therefore the low type has more incentive to deviate, but prefers not to do so provided that (27) holds. Notice that the right-hand side of (27) has a maximum value of $\pi^*(\bar{q}) - \phi$. Therefore (27) is satisfied for at least one price p^p if and only if $\phi \leq \phi_1$, and so only under this condition does a full pooling equilibrium exist. Generically many different pooling prices are compatible with (27), but both types prefer to pool on the equilibrium with $p^p = p^*(\bar{q})$.

Semi-pooling equilibria: In these equilibria the high type reports $r = H$ and charges a price p^{sp}, while the low type randomizes. In particular with probability y the low type pools with the high type, and with probability $1 - y$ separates by reporting $r = L$ and charging

$p^*(L)$. The price p^{sp} and probability of false advertising y must jointly satisfy

$$\pi^*(L) = p^{sp}[1 - G(p^{sp} - q^e)] - \phi\,, \tag{28}$$

where $q^e = \frac{xyL+(1-x)H}{1-x+xy}$ is buyers' Bayesian updated belief about quality after observing a report $r = H$ and price p^{sp}.[23] To understand this condition, note that after reporting $r = H$ and charging p^{sp}, the two types earn respectively $p^{sp}[1-G(p^{sp}-q^e)]-\phi$ and $p^{sp}[1-G(p^{sp}-q^e)]$. However since the low type can also secure $\pi^*(L)$ by reporting truthfully and charging its full-information price, condition (28) is required to ensure that it randomizes.[24] Notice that the right-hand side of (28) has a maximum value of $\pi^*(H) - \phi$, and so a semi-pooling equilibrium exists if and only if $\phi \leq \phi_0$. Generically a continuum of semi-pooling equilibria exist because many values of y and p^{sp} are consistent with condition (28). In particular when $\phi \leq \phi_1$ every $y \in [0, 1]$ is associated with at least one semi-pooling equilibrium, while when $\phi \in (\phi_1, \phi_0]$ every $y \in [0, y^*]$ is associated with at least one semi-pooling equilibrium, where y^* is as defined in Proposition 1. Importantly however, all semi-pooling equilibria are payoff equivalent. This is because in all of them the low type earns $\pi^*(L)$, whilst the high type's payoff $p^{sp}[1 - G(p^{sp} - q^e)]$ always equals $\pi^*(L) + \phi$ as implied from (28).

Separating equilibria: Here, the low type reports $r = L$ and charges $p^*(L)$, whilst the high type reports $r = H$ and charges a price p^s which satisfies

$$\pi^*(L) \leq p^s[1 - G(p^s - H)] \leq \pi^*(L) + \phi \tag{29}$$

To understand this condition, note that in equilibrium buyers perfectly infer quality, and hence the two types earn respectively $\pi^*(L)$ and $p^s[1 - G(p^s - H)]$. The high type could deviate to price $p^*(L)$ (and either report) and receive a payoff of $\pi^*(L)$, so the first inequality is needed to ensure this deviation is unprofitable. The low type could deviate by mimicking the high type and earn $p^s[1 - G(p^s - H)] - \phi$, so the second inequality is needed to ensure this deviation is unprofitable. Notice that a separating equilibrium exists for all values of ϕ because there is always at least one price p^s which satisfies (29). Generically there is a whole continuum. The low type is indifferent over all such equilibria, earning $\pi^*(L)$ in each. The high type prefers to play a least-cost separating equilibrium, which i) for $\phi < \phi_0$ is any price p^s which solves $p^s[1 - G(p^s - H)] = \pi^*(L) + \phi$, and ii) for $\phi \geq \phi_0$ enables it to play its full-information price $p^s = p^*(H)$. Notice that for $\phi < \phi_0$ the high type is forced to engage in price signaling by distorting its price away from the full-information level to prevent the low type from mimicking. Hence in the high type's preferred separating equilibrium it earns $\min\{\pi^*(H), \pi^*(L) + \phi\}$.

[23] Depending on the setting, in the text we sometimes refer to the semi-pooling equilibrium with $y \in \{0, 1\}$ as separating or pooling respectively.

[24] Neither type has an incentive to deviate given condition (28) and appropriate off-path beliefs.

Section C: Remaining Proofs

Proof of Lemma 6. In Section B of the Supplementary Appendix we proved the following. First, pooling equilibria exist if and only if $\phi \leq \phi_1$, and the Pareto dominant pooling equilibrium price equals $p^*(\bar{q})$, which gives respective payoffs $\pi^*(\bar{q}) - \phi$ and $\pi^*(\bar{q})$. Second, semi-pooling equilibria exist if and only if $\phi \leq \phi_0$, take the same form as described in Lemma 6, and always give respective payoffs $\pi^*(L)$ and $\pi^*(L) + \phi$. Third, separating equilibria always exist, and the Pareto dominant separating equilibria give respective payoffs $\pi^*(L)$ and $\min\{\pi^*(H), \pi^*(L) + \phi\}$. Lemma 6 then follows immediately. Note that when $\phi = \phi_1$ the Pareto dominant pooling equilibrium is a special case of the semi-pooling equilibrium with $y = 1$ and $p^{sp} = p^*(\bar{q})$. Also note that when $\phi < \phi_0$ the least-cost separating equilibria are special cases of the semi-pooling equilibria with $y = 0$. Finally when $\phi = \phi_0$ there exists only one semi-pooling equilibrium, and it has $y = 0$ and $p^{sp} = p^*(H)$, such that formally it is the least-cost separating equilibrium. □

Proof of Lemma 7. As a preliminary step, we build on Section B of the Supplementary Appendix to briefly describe the equilibria under asymmetric costs when $c(H) - c(L)$ is close to zero. Firstly, pooling equilibria where both types charge p^p exist if and only if $\phi \leq \phi_1^{c_L}$ where

$$\phi_1^{c_L} \equiv (p^*(\bar{q}, L) - c(L))[1 - G(p^*(\bar{q}, L) - \bar{q})] - \pi^*(L; L) > 0\,. \tag{30}$$

Define Pareto dominant pooling equilibria as follows: pooling equilibria where there is not another pooling equilibrium which gives weakly more to one firm type and strictly more to the other firm type. Then the Pareto dominant pooling equilibria are those with $p^p \in [p^*(\bar{q}, L), p']$ where $p' = \min\{p^*(\bar{q}, H), p''\}$ and p'' is the largest solution to $\phi = (p'' - c(L))[1 - G(p'' - \bar{q})] - \pi^*(L; L)$. The low type earns (weakly) more than $\pi^*(L; L)$ and the high type earns $(p^p - c(H))[1 - G(p^p - \bar{q})]$. Secondly, in a semi-pooling equilibrium the low type mimics the high type by reporting $r = H$ and charging p^{sp} with probability y, where p^{sp} and y satisfy

$$\pi^*(L; L) = (p^{sp} - c(L))[1 - G(p^{sp} - q^e)] - \phi\,, \tag{31}$$

and where $q^e = \frac{xyL + (1-x)H}{xy + 1 - x}$. It can be shown that such an equilibrium exists if $\phi \leq \phi'$ where $\phi' > \phi_1^{c_L}$, and also that at least one equilibrium has $y = 0$. Notice that (31) generically has two solutions since the right-hand side is quasiconcave in p^{sp}, and that as y decreases the larger of these solutions increases. Note also that at $\phi = \phi_1^{c_L}$ the unique pooling equilibrium is part of a continuum of semi-pooling equilibria. Notice further that the high type's payoff $(p^{sp} - c(H))[1 - G(p^{sp} - q^e)]$ can be rewritten with the help of (31) as $\pi^*(L; L) + \phi - (c(H) - c(L))\frac{\pi^*(L;L)+\phi}{p^{sp}-c(L)}$, whereupon it is clear that the high type strictly prefers the semi-pooling equilibrium with $y = 0$ and the largest p^{sp} that solves (31). Define Pareto dominant semi-pooling equilibria as follows: semi-pooling equilibria where there is not another semi-pooling equilibrium which gives weakly more to one firm type and strictly more to the other firm type. Hence for a given $\phi \leq \phi'$, the the Pareto dominant semi-pooling equilibrium has $y = 0$ and the largest p^{sp} that solves (31). Third, it is straightforward to show that the Pareto dominant separating equilibrium (defined in a similar way to above) is least-cost. After defining

$$\phi_0^{c_L} = (p^*(H;H) - c(L))[1 - G(p^*(H;H) - H)] - \pi^*(L;L)\,, \tag{32}$$

one can show that for $\phi < \phi_0^{c_L}$ the least cost separating equilibrium coincides with the Pareto dominant semi-pooling equilibrium, and otherwise for $\phi \geq \phi_0^{c_L}$ has the high type charge $p^*(H;H)$. Note that for small $c(H) - c(L)$ we have $\phi_0^{c_L} > \phi_1^{c_L}$.

Lemma 7 is then proved as follows. Firstly for $\phi \geq \phi_1^{c_L}$ the Pareto dominant equilibrium has least-cost separation as described above. Secondly consider $\phi < \phi_1^{c_L}$. The low type weakly prefers to pool on $p^p \in [p^*(\bar{q}, L), p']$. The high type's pooling payoff is minimized within this set at $p^*(\bar{q}; L)$. Meanwhile given $\phi < \phi_1^{c_L}$ the high type's payoff in a least-cost separating equilibrium is (weakly) less than the case where it charges p_*^{sp} where p_*^{sp} solves equation (31) for $\phi = \phi_1^{c_L}$ and $y = 0$.[25] Hence the high type strictly prefers to pool rather than play a least-cost separating equilibrium provided $\phi < \phi_2^{c_L}$ where

$$\phi_2^{c_L} = (p^*(\bar{q};L) - c(H))[1 - G(p^*(\bar{q};L) - \bar{q})] - \pi^*(L;L) + (c(H) - c(L))[1 - G(p_*^{sp} - H)]\,. \tag{33}$$

Some manipulations show that $\phi_2^{c_L} < \phi_1^{c_L}$.[26] Therefore for $\phi < \phi_2^{c_L}$ Pareto dominant equilibria involve pooling on any of the prices described above, whilst for $\phi \in [\phi_2^{c_L}, \phi_1^{c_L})$ further work is required to determine the Pareto dominant outcome. Thirdly however, notice that as $c(H) \to c(L)$, we have $\phi_0^{c_L} \to \phi_0$, $\phi_1^{c_L}, \phi_2^{c_L} \to \phi_1$, $p^*(\bar{q}, L)$, $p^*(\bar{q}, H) \to p^*(\bar{q})$ and $p^*(q;q) \to p^*(q)$. $\square$

Proof of Remark 4. It suffices to show that $\phi \geq \phi_0$ strictly dominates all $\phi \in [\phi_1, \phi_0)$. Firstly buyer surplus is strictly higher for $\phi \geq \phi_0$, since buyers pay the same price when $q = L$ but a strictly lower price when $q = H$. Secondly consider profits. The low type is indifferent. However the high type prefers $\phi \geq \phi_0$ since its limit payoff is $\pi^*(L) + \phi$ when $\phi \in [\phi_1, \phi_0)$, but $\pi^*(H)$ when $\phi \geq \phi_0$. Thirdly since in both cases there is full separation,

[25] To see this, note that since $\phi < \phi_1^{c_L} < \phi_0^{c_L}$ the largest solution to (31) when $y = 0$ must satisfy $p^{sp} > p^*(H;H)$ and hence the high type is better off when ϕ is larger, since then the largest solution to (31) when $y = 0$ falls closer to $p^*(H;H)$.

[26] This holds provided that $1 - G(p^*(\bar{q}, L) - \bar{q}) > 1 - G(p_*^{sp} - H)$. To prove this, proceed in two steps. First, note that $p_*^{sp} > p^*(\bar{q}, L)$. This is because as shown in the preceding footnote $p_*^{sp} > p^*(H;H)$, and $p^*(H;H) > p^*(H;L) > p^*(\bar{q}, L)$. Second, since $\phi < \phi_1^{c_L}$ we have both $(p^*(\bar{q}, L) - c(L))[1 - G(p^*(\bar{q}, L) - \bar{q})] \geq \pi^*(L;L) + \phi$ and $(p_*^{sp} - c(L))[1 - G(p_*^{sp} - H)] = \pi^*(L;L) + \phi$ which are only compatible if $(p^*(\bar{q}, L) - c(L))[1 - G(p^*(\bar{q}, L) - \bar{q})] \geq (p_*^{sp} - c(L))[1 - G(p_*^{sp} - H)]$. However since $p_*^{sp} > p^*(\bar{q}, L)$ this can only hold if $1 - G(p^*(\bar{q}, L) - \bar{q}) > 1 - G(p_*^{sp} - H)$.

welfare is just the sum of industry profit and buyer surplus, and so it too is maximized by $\phi \geq \phi_0$. $\square$

Proof of Proposition 6. Define $\Delta = xv^*(L) + (1-x)v^*(H) - v^*(xL + (1-x)H)$. Also define $\bar{q}(x) = xL + (1-x)H$. As a preliminary step, note that if $H \leq q^*(L)$ we have $\Delta \geq 0$. Therefore henceforth we assume $H > q^*(L)$, and define $x' \in (0,1)$ such that $\bar{q}(x') = q^*(L)$, and also $\bar{x} \in [x', 1)$ such that $\bar{q}(\bar{x}) = \hat{q}_v$.

First consider how Δ varies with x. Notice that $\Delta(x=0) = \Delta(x=1) = 0$ and

$$\Delta_x(x=1) \geq 0 \iff (H-L)v_q^*(L) - v^*(H) + v^*(L) \geq 0\,. \tag{34}$$

Condition 1 implies that $\Delta_{xx} \geq 0$ for $x \in (0,\bar{x})$, and $\Delta_{xx} < 0$ for $x \in (\bar{x},1)$. Lemma 3 implies that $\Delta(x=x') < 0$. Therefore if (34) holds, $\Delta < 0$ for all $x \in (0,1)$. If (34) does not hold, there exists an $x'' > x'$ satisfying $\Delta(x = x'') = 0$, such that $\Delta < 0$ for $x \in (0, x'')$, and $\Delta > 0$ for $x \in (x'', 1)$. Part i of the proposition then follows immediately, as does the comparative static in $(1-x)$ in part ii of the proposition.

Second consider comparative statics in L. a) The left-hand side of (34) is increasing in L, so if (34) holds initially it also holds for higher values of L. b) Suppose (34) does not hold, in which case

$$\frac{\partial x''}{\partial L} = -x'' \frac{v_q^*(L) - v_q^*(\bar{q}(x''))}{\Delta_x(x = x'')}\,.$$

We need to prove this is positive. Notice that $\Delta_x(x = x'') > 0$ because Δ is single-crossing in x around $x = x''$. Hence we must prove that $v_q^*(\bar{q}(x'')) > v_q^*(L)$. To do this, note that $x'' > x'$ implies $\bar{q}(x'') < q^*(L)$. Under Condition 1i $q^*(L) = \tilde{q}$ and it is then immediate that $v_q^*(\bar{q}(x'')) > v_q^*(L)$. Under Condition 1ii, we know from the proof of Lemma 3 that $v_q^*(q^*(L)) > v_q^*(L)$, such that it is again immediate that $v_q^*(\bar{q}(x'')) > v_q^*(L)$. Hence in both cases the comparative static for L in part ii of the proposition holds.

Third consider comparative statics in H. a) A necessary condition for (34) to hold under Condition 1ii is $v_q^*(L) > v_q^*(H)$. Moreover the derivative of the left-hand side of (34) with respect to H is $v_q^*(L) - v_q^*(H)$, which is positive under Condition 1i since $v_q^*(L) > 0 = v_q^*(H)$, and is also positive under Condition 1ii if (34) holds. Hence if (34) holds initially, it also holds if we increase H. b) Suppose (34) does not hold, in which case

$$\frac{\partial x''}{\partial H} = -(1-x'') \frac{v_q^*(H) - v_q^*(\bar{q}(x''))}{\Delta_x(x = x'')}\,.$$

We need to prove this is positive i.e. prove that $v_q^*(\bar{q}(x'')) > v_q^*(H)$. Again note that $x'' > x'$ implies $\bar{q}(x'') < q^*(L)$. Under Condition 1i $q^*(L) = \tilde{q}$ and it is immediate that $v_q^*(\bar{q}(x'')) > 0 = v_q^*(H)$. Under Condition 1ii some manipulations show that $v_q^*(\bar{q}(x'')) > v_q^*(H)$ must also hold.[27] Hence in both cases the comparative static for H in part ii of the proposition holds. $\square$

Proof of Lemma 8 and Proposition 7. We prove Lemma 8 and Proposition 7 together, in several steps.

1) Given that beliefs depend on the firm's report and not its price, $E(q|r)$ fully determines prices. Hence $\mathbf{y}^*$ is necessary and sufficient to write down expected buyer surplus, total welfare, and profit (before penalties are deducted). Lemma 8i then follows (we return to 8ii later).

2) Buyer surplus. Firstly, buyer surplus is not maximized if any report $r = q_{i<n}$ is sent by more than one type. To see why, consider a new triangular matrix with $y'_{i,i} = \sum_{j=1}^{n-1} y^*_{i,j}$ and $y'_{i,n} = y^*_{i,n}$ for all $i < n$. Strict convexity of $v^*(q) \in (\underset{\sim}{q}, \tilde{q})$ implies that buyer surplus is strictly higher, by Jensen's inequality. Secondly, buyer surplus is not maximized if $E(q|r = q_n) > \tilde{q}$ and $y^*_{i,n} < 1$ for some $i < n$. This is because the derivative of expected buyer surplus with respect to $y^*_{i,n}$ is $x_i [v^*(\tilde{q}) - v^*(q_i)] > 0$. Thirdly, buyer surplus is not maximized if $E(q|r = q_n) = \tilde{q}$, and there exists some $j < k$ such that $y^*_{k,n} < 1$ but $y^*_{j,n} > 0$. To see this, note that $\left.\frac{\partial y^*_{j,n}}{\partial y^*_{k,n}}\right|_{E(q|r=q_n)=\tilde{q}} = -\frac{x_k(\tilde{q}-q_k)}{x_j(\tilde{q}-q_j)}$. The derivative of $E(v)$ with respect to $y^*_{k,n}$, whilst adjusting $y^*_{j,n}$ to ensure $E(q|r = q_n) = \tilde{q}$, is proportional to

$$(\tilde{q} - q_j) [v^*(\tilde{q}) - v^*(q_k)] - (\tilde{q} - q_k) [v^*(\tilde{q}) - v^*(q_j)],$$

which is strictly positive since $v^*(q)$ is strictly convex. Proposition 7i then follows.

3) Profit. Since $\pi^*(q)$ is convex, and strictly so for $q \in (\underset{\sim}{q}, \tilde{q})$, a similar approach to the first part of the previous step shows that expected profit (before penalties are deducted) is maximized by $y^*_{i,i} = 1$ for all i. Hence expected profit once penalties are deducted, is also maximized by $y^*_{i,i} = 1$ for all i, and Proposition 7ii follows.

4) Total welfare. Firstly, total welfare is not maximized if any report $r = q_i$ for $i < n$ is sent by more than one type, and the proof is similar to that for buyer surplus. Secondly, if $E(q|r = q_n) > \tilde{q}$ and there exists some $i < n$ with $y^*_{i,n} < 1$, total welfare is increasing in $y^*_{i,n}$ if and only if $q_i \geq L^*$. To see this, the derivative of $E(TW)$ with respect to $y^*_{i,n}$ is $v^*(\tilde{q}) + a + q_i - v^*(q_i) - \pi^*(q_i)$, which is positive if and only if q_i exceeds a threshold (which we call L^*). Thirdly, total welfare is not maximized if $E(q|r = q_n) = \tilde{q}$, and there exists some $j < k$ such that $y^*_{k,n} < 1$ but $y^*_{j,n} > 0$. The proof closely follows the same arguments for buyer surplus. Proposition 7iii then follows.

5) Implementation. Note that the maximum gain from false advertising is $\bar{\phi} = \pi^*(q_n) - \pi^*(q_1)$.

[27] This is immediate if $\bar{q}(x'') \geq \hat{q}_v$. Suppose instead that $\bar{q}(x'') < \hat{q}_v$. If in fact $v^*_q(\bar{q}(x'')) \leq v^*_q(H)$ this would imply $v^*(H) \geq v^*(\bar{q}(x'')) + (H - \bar{q}(x''))v^*_q(\bar{q}(x''))$. However since $(1 - x'')(H - \bar{q}(x'')) = x''(\bar{q}(x'') - L)$, this in turn would imply $\Delta(x = x'') \geq x'' \left[(\bar{q}(x'') - L)v^*_q(\bar{q}(x'')) + v^*(L) - v^*(\bar{q}(x''))\right]$, which is strictly positive given the properties of $v^*(q)$. Hence, this yields a contradiction since $\Delta(x = x'') = 0$ by definition .

First, set $\phi(q_i, q_j) = \bar{\phi}$ for all $j \notin \{q_i, q_n\}$ so that in any equilibrium, each firm either reports truthfully or reports $r = q_n$. Second, for any type i for whom $y^*_{i,i} = 1$, also set $\phi(q_i, q_n) = \bar{\phi}$. Third, for any type i for whom $y^*_{i,n} = 1$, set $\phi(q_i, q_n) = 0$. Fourth, let $q^e_n = (\sum_{j=1}^n x_j y^*_{j,n} q_j)/(\sum_{j=1}^n x_j y^*_{j,n})$. For any type i for whom $y^*_{i,i} = 1 - y^*_{i,n}$ and $y^*_{i,n} \in (0, 1)$ (there is at most one such i) set $\phi(q_i, q_n) = \pi^*(q^e_n) - \pi^*(q_i)$. Fifth, it is easy to see there is a unique equilibrium outcome in which $\mathbf{y}^*$ is played, and so Lemma 8ii follows. $\square$

Proof of Lemma 9. Profit is quasiconcave in p because the hazard rate is increasing. (i) When $q \leq \utilde{q}^m$ no price above marginal cost can attract any demand and so the market is inactive. (ii) Suppose $q > \utilde{q}^m$. An interior solution must satisfy the following first order condition

$$\frac{1 - F(p/q)}{f(p/q)} - \left(\frac{p - c}{q}\right) = 0\,. \tag{35}$$

Notice that the left-hand side is decreasing in p/q, and so there exists an interior solution if the left-hand side is strictly positive when evaluated at $p/q = \underline{\theta}$, and strictly negative at $p/q = \bar{\theta}$. In addition to $q > \utilde{q}_m$, this requires $1/f(\underline{\theta}) - \underline{\theta} + \frac{c}{q} > 0$. The former necessarily holds if $\underline{\theta} f(\underline{\theta}) \leq 1$; if instead $\underline{\theta} f(\underline{\theta}) > 1$, it holds if $q < \tilde{q}^m \equiv cf(\underline{\theta})/[\underline{\theta} f(\underline{\theta}) - 1]$. (iii) If $\underline{\theta} f(\underline{\theta}) > 1$ and $q \geq \tilde{q}^m$ then a corner solution arises as the first derivative of profit is negative for all $p/q \in [\underline{\theta}, \bar{\theta}]$. At this point, the optimal price equals the willingness-to-pay of the marginal buyer $\underline{\theta} q$. $\square$

Proof of Lemma 10. Observe that for any $\theta \in [\underline{\theta}, \bar{\theta}]$ demand curvature is the same under both F and G. First note that the market can now be covered if and only if $\tilde{q}^m < \infty$ or $\underline{\theta} f(\underline{\theta}) > 1$, which is equivalent to $\underline{\theta} g(\underline{\theta})/[1 - G(\underline{\theta})] > 1$. The inequality holds i) at $\underline{\theta} = b$ for any distribution with $b < \infty$ since $g(b) > 0$ by assumption, and ii) as $\underline{\theta} \to b$ for any distribution with $b = \infty$ since the hazard rate is strictly positive and weakly increasing. Notice also that the left-hand side is strictly increasing in $\underline{\theta}$ due to the increasing hazard rate. Hence the market is covered for sufficiently high $\underline{\theta}$. Second simple manipulations show that $v^*_{qq}(q) > 0$ for all $q \in (\utilde{q}^m, \tilde{q}^m)$ if and only if $z_v(\theta) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, where $z_v(\theta)$ is defined in the Supplementary Appendix. Moreover Condition 1 ensures the latter holds provided that $\underline{\theta}$ is sufficiently high. Third, it is simple to verify that $v^*_q(q) = E(\theta) - \underline{\theta}$ for $q > \hat{q}^m_v$, which is strictly less than $\lim_{q \uparrow \hat{q}^m_v} v^*_q(q)$ provided that $\lim_{\varepsilon \to \underline{\theta}} \sigma(\varepsilon) > -\infty$. $\square$

Proof of Proposition 8. The proof follows easily from earlier arguments and is therefore omitted. $\square$

References

[1] Duembgen L. (2010) “Bounding Standard Gaussian Tail Probabilities” arXiv:1012.2063

[2] Fabinger M. and G. Weyl (2016) “The Average-Marginal Relationship and Tractable Equilibrium Forms” Working paper

[3] Weyl G. and J. Tirole (2012) “Market Power Screens Willingness-to-Pay” Quarterly Journal of Economics, 127, 1971-2003